Stochastic PDEs with Random Set Coefficients

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Abstract

This contribution addresses stochastic PDEs with random set coefficients. A typical example is the elliptic PDE

$$-\operatorname{div}\big(A(x)\operatorname{grad}u(x)\big) = f(x)$$

where the excitation and the coefficient matrix are given by any of the following: (a) a random field (a stochastic process with respect to the spatial variable); (b) a random set; (c) a random field whose parameters are random sets; (d) a combination thereof. As soon as random sets and stochastic processes are involved, the solution u is a set-valued process. The question arises in what sense it can be viewed as a random set.

For a stationary, Gaussian random field A it suffices to specify the expectation values $\mu \equiv \mathrm{E}(A(x))$ and the autocovariance function $C(\rho) = \mathrm{COV}(A(x), A(y))$ which then depends only on the distance $\rho = |x - y|$. As a starting point, we consider a parametrized autocovariance function of the form $C(\rho) = \sigma^2 \exp\big(-|\rho|/L\big)$ with the field variance σ^2 and the correlation length L as parameters. A useful feature of this type of random field is that it can be obtained as solution to the Langevin equation, W_t denoting Wiener process,

$$\mathrm{d}X_t = -\tfrac{1}{L}X_t + \sqrt{\tfrac{2}{L}}\,\sigma\,\mathrm{d}W_t, \quad X_0 \sim \mathcal{N}(0,\sigma^2). \tag{1}$$

A random set is a map X which assigns to every ω from a probability space (Ω, Σ, P) a subset $X(\omega)$ of a target space $\mathbb{E}$ such that the upper inverses $X^-(B) = \{\omega \in \Omega : X(\omega) \cap B \neq \emptyset\}$ are measurable for every Borel subset B of $\mathbb{E}$. An important tool is the *fundamental measurability theorem* that states (if $\mathbb{E}$ is a Polish space) the equivalence of the defining measurability property of $X^-(B)$ for Borel, open, and closed subsets B as well as the equivalence with the existence of a *Castaing representation.* A set-valued random variable such that $X^-(B)$ is measurable for every open set B is called *Effros-measurable.* Starting from a random field whose correlation length, e.g., is an interval, the assignment

$$\omega \to \{A_L(x,\omega) : L \in [\underline{L}, \overline{L}]\},$$

where x is a point in space and $A_L(x,\omega)$ is a realization at point x of the field with correlation length L, defines a random set. It is the purpose of this contribution to present a proof of this fact. Thanks to the representation (1), the continuity of the map $L \to A_L(x,\omega)$ can be derived from the results of [1, 2]. From there, a Castaing representation can be immediately obtained, which leads to the Effros measurability; the fundamental measurability theorem completes the argument. The methods will be demonstrated at the hand of a numerical example, employing polynomial chaos expansion as a computational device.

Keywords. Random fields, random sets, set-valued stochastic processes.

References

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