New Results on the Inferential Complexity of Credal Networks

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Abstract

Credal networks are graph-based multivariate statistical models where irrelevance assessments between sets of variables are concisely described by means of an *acyclic directed graph* whose nodes are identified with variables. Here we focus on categorical variables [Cozman, 2000]. A credal network encodes a set of *Markov conditions*: the non-descendant non-parents of any variable are irrelevant to it once the value of the parents is known. The complete specification of a credal network requires the quantification of local conditional credal sets, closed and convex sets of conditional probability distributions [Levi, 1980]. Credal networks represent a joint credal set over all variables in the model, and thus allow for the distinction of randomness and ignorance, and facilitate the elicitation of the parameters from experts [Antonucci et al., 2007, Antonucci et al., 2009, Piatti et al., 2010].

To fully characterize the credal set induced by a credal network we need to settle on the concept of irrelevance adopted, and thus on the semantics of the arcs in the graph. The most commonly used concepts in the literature are *strong independence* and *epistemic irrelevance*. The former states that two variables X and Y are strongly independent if the joint credal set of X, Y can be regarded as originating from a number of precise models in each of which the two variables are stochastically independent. Strong independence is closely related to the sensitivity analysis interpretation of credal sets, which regards an imprecise model as arising out of partial ignorance of a precise one [Antonucci and Piatti, 2009, Zaffalon and Miranda, 2009]. A variable X is epistemically irrelevant to a variable Y if observing X does not affect our beliefs about Y. In other words, by making an epistemic irrelevance assessment, we are stating that our beliefs about Y do not change after receiving information about X [Walley, 1991].

Usually, credal networks are used to compute tight bounds on the expectation of some variable conditional on the value of some other variables, a task we call *predictive inference*. The complexity of this task varies greatly according to the topology of the underlying digraph and the irrelevance concept adopted. For instance, the 2U algorithm of [Fagiuoli and Zaffalon, 1998] can solve the problem in polynomial time if the digraph is a polytree, variables are binary and strong independence is assumed. When instead epistemic irrelevance is adopted, no analogous polynomial-time algorithm for the task is known. On the other hand, [de Cooman et al., 2010] developed a polynomial-time algorithm for predictive inferences in epistemic trees, that is, credal trees under epistemic irrelevance. No such algorithm is known to exist under strong independence.

The aim of this work is to present the following three new contributions that appeared in [Mauá et al., 2013], and to report on the current state of knowledge of the theoretical computational complexity of predictive inference in credal networks, as summarized in Table 1 (new results appear in boldface).

- Epistemic irrelevance and strong independence induce the same upper and lower predictive probabilities for the last (in topological order) hidden node in HMM-like credal trees. This implies that we can use the algorithm of [de Cooman et al., 2010] for credal trees under epistemic irrelevance to compute tight bounds on the posterior expectation of the last hidden node also under strong independence.

- Predictive inferences under strong independence in credal trees are NP-hard even if all variables are ternary, which shows that is unlikely that polynomial-time algorithms for these networks exist, in striking difference with the case of epistemic irrelevance.

Table 1: Summary of the computational complexity of predictive inference in credal networks.

Model	Strong	Epistemic
HMM	**P**	P
Tree	**NP-hard**	P
Polytree	NP-hard	**NP-hard**
General	NP^{PP}-hard	$\mathbf{NP^{PP}}$**-hard**

- Predictive inference in networks where root nodes are vacuous and the remaining ones are precise is invariant to the irrelevance concept used. This in turn implies that the task in credal polytrees under epistemic irrelevance is NP-hard, even if all variables are at most ternary, as this is the case under strong independence unless all variables are binary [Fagiuoli and Zaffalon, 1998, de Campos and Cozman, 2005].

Keywords. Credal networks, graphical models, epistemic irrelevance, strong independence, coherent lower prevision, credal sets.

References

[Antonucci et al., 2009] Antonucci, A., Brühlmann, R., Piatti, A., and Zaffalon, M. (2009). Credal networks for military identification problems. *International Journal of Approximate Reasoning*, 50(4):666–679.

[Antonucci and Piatti, 2009] Antonucci, A. and Piatti, A. (2009). Modeling unreliable observations in Bayesian networks by credal networks. In Godo, L. and Pugliese, A., editors, *Scalable Uncertainty Management, Third International Conference, SUM 2009, Washington, DC, USA, September 28–30, 2009. Proceedings*, volume 5785 of *Lecture Notes in Computer Science*, pages 28–39. Springer.

[Antonucci et al., 2007] Antonucci, A., Piatti, A., and Zaffalon, M. (2007). Credal networks for operational risk measurement and management. In *International Conference on Knowledge-Based and Intelligent Information & Engineering Systems (KES2007)*.

[Cozman, 2000] Cozman, F. G. (2000). Credal networks. *Artificial Intelligence*, 120(2):199–233.

[de Campos and Cozman, 2005] de Campos, C. P. and Cozman, F. G. (2005). The inferential complexity of bayesian and credal networks. In *Proceedings of the 19th international joint conference on Artificial intelligence*, IJCAI'05, pages 1313–1318, San Francisco, CA, USA. Morgan Kaufmann Publishers Inc.

[de Cooman et al., 2010] de Cooman, G., Hermans, F., Antonucci, A., and Zaffalon, M. (2010). Epistemic irrelevance in credal nets: the case of imprecise markov trees. *International Journal of Approximate Reasoning*, 51(9):1029–1052.

[Fagiuoli and Zaffalon, 1998] Fagiuoli, E. and Zaffalon, M. (1998). 2U: An exact interval propagation algorithm for polytrees with binary variables. *Artificial Intelligence*, 106(1):77–107.

[Levi, 1980] Levi, I. (1980). *The Enterprise of Knowledge.* MIT Press, London.

[Mauá et al., 2013] Mauá, D. D., de Campos, C. P., Benavoli, A., and Antonucci, A. (2013). On the complexity of strong and epistemic credal networks. Technical report, Dalle Molle Institute for Artificial Intelligence. IDSIA-02-13, http://www.idsia.ch/idsiareport/IDSIA-02-13.pdf.

[Piatti et al., 2010] Piatti, A., Antonucci, A., and Zaffalon, M. (2010). *Building Knowledge-Based Systems by Credal Networks: A Tutorial.* Nova Science.

[Walley, 1991] Walley, P. (1991). *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, London.

[Zaffalon and Miranda, 2009] Zaffalon, M. and Miranda, E. (2009). Conservative inference rule for uncertain reasoning under incompleteness. *Journal of Artificial Intelligence Research*, 34:757–821.