Coherent updating of 2-monotone previsions

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Abstract

The conditions for a 2-monotone lower prevision to be uniquely updated to a conditional lower prevision are determined. Then a number of particular cases are investigated: completely monotone lower previsions, for which equivalent conditions in terms of the focal elements of the associated belief function are established; random sets, for which some conditions in terms of the measurable selections can be given; and minitive lower previsions, that are shown to correspond to the particular case of vacuous lower previsions.

Keywords. Coherent lower previsions, n-monotonicity, belief functions, minitive measures, natural extension, regular extension.

1 Introduction

The theory of imprecise probabilities contains a wide variety of mathematical models that are of interest in situations where it is unfeasible to determine the probability model associated to an experiment with certain guarantees. Under any of them, one important problem is that of updating the model under the light of new information. Unfortunately, this problem is far from settled, and quite a number of different rules have been proposed. Out of them, arguably some of the most popular are Dempster's rule of conditioning [11], regular extension [4] and natural extension [27].

In order to be able to choose one rule above the others, it is essential to have a clear interpretation of the mathematical model we are using. In this paper, we shall consider the behavioural approach championed by Peter Walley [27], that has its roots in the works on subjective probability by Bruno de Finetti [10]. This approach regards lower and upper probabilities as supremum and infimum betting rates, and focuses on a consistency notion between these betting rates called *coherence*.

When we move to the conditional case, there is also a notion of coherence that tells us if the conditional betting rates are compatible with the unconditional ones. However, this notion does not suffice to uniquely determine the conditional models from the unconditional ones. This was showed for instance in [20], where it was established that in general we may have an infinite number of conditional models compatible with the unconditional one, and that the smallest and greatest such models are determined by the procedures called *natural* and *regular* extension, respectively. In this paper, we investigate under which conditions there is only one conditional model that is coherent with the unconditional one.

Walley's theory is established in terms of lower and upper *previsions* (or expectations), because these are more informative than the lower and upper probabilities that can be considered as a particular case. We shall recall the basics from the theory of coherent lower previsions in Section 2. Then we shall focus on a particular case of coherent lower previsions: those satisfying the property of 2-monotonicity [2, 7]. Lower previsions with this property have the advantage of being uniquely determined by their restrictions to events (a 2-monotone lower probability) by means of the Choquet integral.

After establishing a necessary and sufficient condition for the uniqueness of the coherent extensions to the conditional case in Section 3, we focus on two particular cases of 2-monotone lower previsions. First, in Section 4 we consider completely monotone lower previsions, that correspond to the Choquet integral with respect to a belief function [7]; then we discuss minimum-preserving lower previsions in Section 5. Our results in this section illustrate one interesting fact: that the coherence between unconditional and conditional lower probabilities studied in [30] is not equivalent to the coherence of the respective lower previsions they determine by means of the Choquet integral.

Due to limitations of space, proofs have been omitted.

2 Preliminary concepts

2.1 Coherent lower previsions

Consider a possibility space Ω, that we shall assume in this paper to be *finite*. A *gamble* is a real-valued functional defined on Ω. We shall denote by $\mathcal{L}(\Omega)$ the set of all gambles on Ω. One instance of gambles are the indicators of events. Given a subset A of Ω, the indicator function of A is the gamble that takes the value 1 on the elements of A and 0 elsewhere. We shall denote this gamble by I_A, or by A when no confusion is possible.

A *lower prevision* is a functional $\underline{P}$ defined on a set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. Given a gamble f, $\underline{P}(f)$ is understood to represent a subject's supremum acceptable buying price for f, in the sense that for any $\epsilon > 0$ the transaction $f - \underline{P}(f) + \epsilon$ is acceptable for him.

Using this interpretation, we can derive a notion of coherence:

Definition 1. A lower prevision $\underline{P} : \mathcal{L}(\Omega) \to \mathbb{R}$ is called *coherent* if and only if it satisfies the following properties for every $f, g \in \mathcal{L}(\Omega)$ and every $\lambda > 0$:

(C1) $\underline{P}(f) \geq \min f$.

(C2) $\underline{P}(\lambda f) = \lambda \underline{P}(f)$.

(C3) $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g)$.

The interpretation of this notion is that the acceptable buying prices encompassed by $\{\underline{P}(f) : f \in \mathcal{L}(\Omega)\}$ are consistent with each other. In the particular case when $\underline{P}$ satisfies (C3) with equality for every $f, g \in \mathcal{L}(\Omega)$, it is called a *linear* prevision. Any coherent lower prevision is the *lower envelope* of the set of linear previsions that dominate it, i.e.,

$$\underline{P}(f) = \min\{P(f) : P \text{ linear prevision}, P \geq \underline{P}\}.$$

The conjugate functional $\overline{P}$ of a coherent lower prevision $\underline{P}$, given by $\overline{P}(f) = -\underline{P}(-f)$ for every $f \in \mathcal{L}(\Omega)$, is called a coherent *upper* prevision. It corresponds to the upper envelope of the set of linear previsions that dominate $\underline{P}$.

A coherent lower prevision defined only on indicators of events is called a *coherent lower probability*. In particular, the restriction of a linear prevision to indicators of events corresponds to a (finitely additive) probability measure. Hence, coherent lower previsions are simply lower envelopes of closed and convex sets of probability measures, and as such they can also be given a Bayesian sensitivity analysis interpretation.

One particular case of coherent lower previsions are the *vacuous* ones. They correspond to the case where we have the information that the outcome of the experiment belongs to some set A (and nothing else). In that case, our coherent lower prevision is given by

$$\underline{P}(f) = \min_{\omega \in A} f(\omega) \ \forall f \in \mathcal{L}(\Omega). \tag{1}$$

Although a linear prevision is uniquely determined by the probability measure that is its restriction to events, this is not the case for lower previsions: a coherent lower probability will have in general more than one coherent extension to the set of all gambles. This is the reason why the theory is established in terms of gambles instead of events. Interestingly, there are some cases where the restriction to events uniquely determines the coherent lower prevision. One particular case that shall be important in this paper is that where the restriction to events is 0–1-valued:

Lemma 1. *[27, Note 4, Section 3.2.6] Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$ whose restriction to events is 0–1-valued. Then $\underline{P}$ is the unique coherent extension of its restriction to events, and it is given by*

$$\underline{P}(f) = \sup_{F : \underline{P}(F)=1} \inf_{\omega \in F} f(\omega);$$

moreover, the class $\{F \subseteq \Omega : \underline{P}(F) = 1\}$ is a filter.

This applies in particular for the vacuous lower previsions in Eq. (1).

2.2 Conditional lower previsions

Given a partition $\mathcal{B}$ of the possibility space Ω, a *conditional lower prevision* on $\mathcal{L}(\Omega)$ is a functional $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}(\Omega)$ that to any gamble f and any $B \in \mathcal{B}$ assigns the value $\underline{P}(f|B)$, that represents a subject's supremum acceptable buying price for f, if he comes to know later that the outcome of the experiment belongs to the subset B of Ω. Thus, $\underline{P}(\cdot|B)$ is a functional on $\mathcal{L}(\Omega)$ for every $B \in \mathcal{B}$. By putting all these values together, we end up with the gamble

$$\underline{P}(f|\mathcal{B}) := \sum_{B \in \mathcal{B}} I_B(f - \underline{P}(f|B)).$$

Similarly to conditions (C1)–(C3), we can establish a notion of coherence for conditional lower previsions.

Definition 2. A conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}(\Omega)$ is *separately coherent* when

(SC1) $\underline{P}(f|B) \geq \min_{\omega \in B} f(\omega)$,

(SC2) $\underline{P}(\lambda f|B) = \lambda \underline{P}(f|B)$,

(SC3) $\underline{P}(f + g|B) \geq \underline{P}(f|B) + \underline{P}(g|B)$

for every $f, g \in \mathcal{L}(\Omega), \lambda > 0$ and $B \in \mathcal{B}$.

The behavioural interpretation of this notion is that the acceptable conditional buying prices encompassed by $\underline{P}(\cdot|B)$ are consistent with each other for every fixed set B in the partition $\mathcal{B}$. Together they imply $\underline{P}(B|B) = 1 \ \forall B \in \mathcal{B}$.

If we start with a coherent lower prevision $\underline{P}$ and consider a partition $\mathcal{B}$ of the space Ω, there is in general not a unique way of updating $\underline{P}$ into a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$. This is related to the problem of conditioning on sets of probability zero, which has attracted a lot of attention in the literature [3, 13, 18]; see also [27, Chapter 6] for the approach considered in this paper. In the next section we detail how the conditional lower prevision may be derived and we formulate the problem we shall study in this paper.

2.3 Formulation of the problem

Consider now a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\Omega)$, let $\mathcal{B}$ be a partition of Ω and assume we want to update $\underline{P}$ into a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}(\Omega)$.

One strategy to derive $\underline{P}(\cdot|\mathcal{B})$ from $\underline{P}$ is to verify that the assessments present in these two lower previsions are compatible with each other. This gives rise to the concept of *joint coherence*, which is studied in much detail in [27, Chapters 6 and 7]. In this case, where we are dealing with finite spaces, we have the following characterisation:

Proposition 1. *[27, Theorem 6.5.4] Consider a coherent lower prevision $\underline{P}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}(\Omega)$, where Ω is a finite space. They are jointly coherent when*

$$\underline{P}(B(f - \underline{P}(f|B))) = 0 \ \forall f \in \mathcal{L}(\Omega), B \in \mathcal{B}. \quad (2)$$

The above equation is called the *Generalised Bayes Rule*, because it reduces to the well-known Bayes' rule in the precise case. It holds trivially when $\overline{P}(B) = 0$, so any conditional lower prevision $\underline{P}(\cdot|B)$ is compatible with $\underline{P}$ in that case; on the other hand, if $\underline{P}(B) > 0$ then for every gamble f there is a unique real number μ such that $\underline{P}(B(f - \mu)) = 0$, so there is only one conditional lower prevision $\underline{P}(\cdot|B)$ that is compatible with $\underline{P}$.

The most interesting case is that where the conditioning event has zero lower probability and positive upper probability, i.e., that of $\underline{P}(B) = 0 < \overline{P}(B)$. In that case, there is usually an infinite number of conditional lower previsions that are compatible with $\underline{P}$; there were characterised in [20], where it was proven that they are bounded by the so-called natural and regular extensions.

Definition 3. Given $B \in \mathcal{B}$, the *natural extension* $\underline{E}(\cdot|B)$ induced by $\underline{P}$ is given by:

$$\underline{E}(f|B) := \begin{cases} \inf_{P \geq \underline{P}} \{P(f|B)\} & \text{if } \underline{P}(B) > 0 \\ \min_{\omega \in B} f(\omega) & \text{otherwise} \end{cases}$$

for any gamble $f \in \mathcal{L}(\Omega)$.

The natural extension is vacuous when the conditioning event has zero lower probability, and is uniquely determined by Eq. (2) otherwise. Although it produces a conditional lower prevision that is coherent with $\underline{P}$, it is arguably too uninformative. A more informative alternative is called the regular extension:

Definition 4. Given $B \in \mathcal{B}$, the *regular extension* $\underline{R}(\cdot|B)$ induced by $\underline{P}$ is given by:

$$\underline{R}(f|B) := \begin{cases} \inf_{P(B)>0, P \geq \underline{P}} \{P(f|B)\} & \text{if } \overline{P}(B) > 0 \\ \min_{\omega \in B} f(\omega) & \text{otherwise} \end{cases}$$

for any gamble $f \in \mathcal{L}(\Omega)$.

Hence, regular extension corresponds to applying Bayes' rule whenever possible on the set of precise models compatible with our conditional lower prevision, and to take then the lower prevision of the resulting set of conditional previsions. It has been proposed as an updating rule in a number of works in the literature [4, 8, 14, 15, 17, 28].

It turns out that the natural and the regular extensions characterise the set of conditional lower previsions that are jointly coherent with $\underline{P}$:

Proposition 2. *[20, Theorem 9] Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$ and $\mathcal{B}$ a partition of Ω such that $\overline{P}(B) > 0$ for any $B \in \mathcal{B}$. Then a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ is coherent with $\underline{P}$ if and only if $\underline{P}(f|B) \in [\underline{E}(f|B), \underline{R}(f|B)]$ for every $f \in \mathcal{L}(\Omega)$ and every $B \in \mathcal{B}$.*

In this paper we shall not deal with the case $\overline{P}(B) = 0$ because then any conditional model $\underline{P}(\cdot|B)$ satisfies the Generalised Bayes Rule with $\underline{P}$.

The conditional lower previsions determined by the natural and regular extension may not coincide when $\underline{P}(B) = 0 < \overline{P}(B)$ (see for instance Example 2 later on). In this paper, we are going to characterise their equality for one interesting particular case of coherent lower previsions: the 2-monotone ones. As particular cases, we shall consider completely monotone lower previsions, random sets and possibility measures.

3 Updating 2-monotone lower previsions

One important instance of coherent lower previsions are the n-monotone ones, that were first introduced by Choquet in [2]:

Definition 5. A coherent lower prevision $\underline{P}$ on $\mathcal{L}(\Omega)$ is called *n-monotone* if and only if

$$\underline{P}\left(\bigvee_{i=1}^{p} f_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,p\}} (-1)^{|I|+1} \underline{P}\left(\bigwedge_{i \in I} f_i\right) \quad (3)$$

for all $2 \leq p \leq n$, and all $f_1, \dots, f_p$ in $\mathcal{L}(\Omega)$, where $\vee$ denotes the point-wise maximum and $\wedge$ the point-wise minimum.

In particular, a coherent lower probability $\underline{P} : \mathcal{P}(\Omega) \to [0,1]$ is n-monotone when

$$\underline{P}\left(\bigcup_{i=1}^{p} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,p\}} (-1)^{|I|+1} \underline{P}\left(\bigcap_{i \in I} A_i\right) \quad (4)$$

for all $2 \leq p \leq n$, and all subsets $A_1, \dots, A_p$ of Ω.

Although a coherent lower prevision is not determined uniquely by its restriction to events, it is when we require in addition the property of n-monotonicity, in the following sense: given a n-monotone lower probability, its natural extension is the only n-monotone extension to $\mathcal{L}(\Omega)$. It corresponds moreover to the Choquet integral [12] with respect to this fuzzy measure [7, 26], so we have that

$$\underline{P}(f) := (C) \int f d\underline{P} = \inf f + \int_{\inf f}^{\sup f} \underline{P}(f \geq t) dt$$

for every gamble f.

A coherent lower prevision on $\mathcal{L}(\Omega)$ that is n-monotone for all $n \in \mathbb{N}$ is called *completely monotone*, and its restriction to events is a *belief function*; its conjugate $\overline{P}$ is a plausibility function. One example of completely monotone coherent lower previsions are the vacuous ones in Eq. (1); another one is given by the linear previsions, that moreover satisfy Eq. (3) with equality for every n.

In particular, a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\Omega)$ is 2-monotone if and only if it satisfies Eq. (3) for $n = 2$, that is, if and only if

$$\underline{P}(f \vee g) + \underline{P}(f \wedge g) \geq \underline{P}(f) + \underline{P}(g)$$

for every $f, g \in \mathcal{L}(\Omega)$. On the other hand, we deduce from Eq. (4) that a coherent lower probability on $\mathcal{P}(\Omega)$ is called 2-monotone whenever

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \ \forall A, B \subseteq \Omega.$$

In this section, we are going to determine under which conditions a 2-monotone lower prevision $\underline{P}$ on $\mathcal{L}(\Omega)$ can be uniquely updated to a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ that is coherent with $\underline{P}$, in the sense of Eq. (2). In order to do this, we shall use the formula for the conditional lower probability determined by regular extension:

Proposition 3. *[26, Theorem 7.2] Let $\underline{P}$ be a 2-monotone lower prevision on $\mathcal{L}(\Omega)$, and consider $B \subseteq \Omega$ such that $\overline{P}(B) > 0$. Then for any event A,*

$$\underline{R}(A|B) = \begin{cases} \frac{\underline{P}(A\cap B)}{\underline{P}(A\cap B)+\overline{P}(A^c\cap B)} & \text{if } \overline{P}(A^c \cap B) > 0, \\ 1 & \text{otherwise,} \end{cases} \quad (5)$$

and $\underline{R}(\cdot|B)$ is a 2-monotone lower probability.

Interestingly, we shall show in Example 3 later on that in general $\underline{R}(\cdot|B)$ need not be 2-monotone on gambles. As we shall see, we can only guarantee 2-monotonicity on gambles when the conditioning event has zero lower probability and positive upper probability.

To see that Eq. (5) does not hold without the assumption of 2-monotonicity, consider the following example:

Example 1. Consider $\Omega = \{a, b, c, d\}$ and let P_1, P_2 be the linear previsions determined by the mass functions p_1, p_2 given by

	a	b	c	d
p_1	0.5	0.5	0	0
p_2	0.25	0.25	0.25	0.25

It has been showed in [26, Section 6] that the lower envelope $\underline{P}$ of $\{P_1, P_2\}$ is a coherent lower prevision that is not 2-monotone. Consider $B = \{a, b\}$ and $A = \{a\}$. Then $\overline{P}(A^c \cap B) = \overline{P}(\{b\}) = 0.5 > 0$, and

$$\frac{\underline{P}(A \cap B)}{\underline{P}(A \cap B) + \overline{P}(A^c \cap B)} = \frac{0.25}{0.25 + 0.5} = \frac{1}{3};$$

on the other hand any $P \geq \underline{P}$ is given by $\alpha P_1 + (1-\alpha)P_2$, where $\alpha \in [0,1]$; since $P_1(\{a\}) = P_1(\{b\})$ and $P_2(\{a\}) = P_2(\{b\})$, it follows that any $P \geq \underline{P}$ must satisfy $P(\{a\}) = P(\{b\})$ too, whence $\underline{R}(A|B) = 0.5$. Hence, Eq. (5) does not hold. ♦

From Proposition 3 we deduce the following:

Proposition 4. *Let $\underline{P}$ be a 2-monotone lower prevision on $\mathcal{L}(\Omega)$ and consider $B \subseteq \Omega$ such that $\underline{P}(B) = 0 < \overline{P}(B)$. Then for any gamble f*

$$\underline{R}(f|B) = \min_{\omega \in C} f(\omega),$$

where C is the smallest subset of B satisfying $\underline{R}(C|B) = 1$.

Interestingly, this shows that, if the lower prevision $\underline{P}$ satisfies 2-monotonicity, when the conditioning event B has zero lower probability and positive upper probability, the regular extension $\underline{R}(\cdot|B)$ is a completely monotone lower prevision, even if the lower prevision $\underline{P}$ we start from is not completely monotone.

Using these results, we can determine in which cases the natural and regular extensions coincide:

Proposition 5. *Let $\underline{P}$ be a 2-monotone lower prevision on $\mathcal{L}(\Omega)$, and consider $B \subseteq \Omega$ with $\overline{P}(B) > 0 = \underline{P}(B)$. The following are equivalent:*

1. *$\underline{E}(f|B) = \underline{R}(f|B)$ for every $f \in \mathcal{L}(\Omega)$.*
2. *$\underline{E}(A|B) = \underline{R}(A|B)$ for every $A \subseteq \Omega$.*
3. *$\overline{P}(\{\omega\}) > 0$ for every $\omega \in B$.*

We immediately deduce the following:

Theorem 1. *Let $\underline{P}$ be a 2-monotone lower prevision on $\mathcal{L}(\Omega)$, and let $\mathcal{B}$ be a partition of Ω. Then $\underline{E}(\cdot|\mathcal{B}) = \underline{R}(\cdot|\mathcal{B})$ if and only if $\overline{P}(\{\omega\}) > 0\ \forall \omega \in B \subseteq \Omega$ s.t. $\underline{P}(B) = 0 < \overline{P}(B)$.*

To see that this result cannot be extended to arbitrary coherent lower previsions, it suffices to consider the coherent lower prevision $\underline{P}$ in Example 1, $B = \{c, d\}$ and $A = \{c\}$: we get $\underline{E}(A|B) = 0 < 0.5 = \underline{R}(A|B)$.

4 Coherent updating of completely monotone lower previsions

We consider next the case where the lower prevision $\underline{P}$ on $\mathcal{L}(\Omega)$ is completely monotone.

One of the most important rules in that case is Dempster's rule of conditioning [11, 24], where, given a plausibility function $\overline{P}$ on $\mathcal{P}(\Omega)$ and a conditioning event B with $\overline{P}(B) > 0$, the conditional plausibility is defined by

$$\overline{P}(A|B) := \frac{\overline{P}(A \cap B)}{\overline{P}(B)}.$$

However, this conditional upper probability is not coherent with the unconditional upper probability $\overline{P}$ [31]; see also [27, Section 5.13] and [29]. Thus, Dempster's rule is not interesting from the behavioural point of view, and we shall focus in this section on the natural and the regular extensions instead.

Given a conditioning event B with $\overline{P}(B) > 0$, its regular extension is determined by Eq. (5). This formula has also been established in a few papers ([14, Theorem 3.4]; [15, Proposition 4]; see also [4, 11]). Moreover, it has been established in [14, 15, 25] that the restriction of $\underline{R}(\cdot|B)$ to events is a belief function for every $B \subseteq \Omega$ such that $\underline{P}(B) > 0$.

The equality between the natural and the regular extensions of $\underline{P}$ is characterised by Theorem 1. In this section, we give equivalent conditions in terms of the focal elements of $\underline{P}$.

Definition 6. [24] Given a belief function $\underline{P}$ on $\mathcal{P}(\Omega)$, its *Möbius inverse* $m : \mathcal{P}(\Omega) \to [0, 1]$ is given by

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(B)\ \forall A \subseteq \Omega.$$

It holds that $\underline{P}(A) = \sum_{B \subseteq A} m(B)$, and m is called a *basic probability assignment* within the evidential theory of Shafer. For the plausibility function $\overline{P}$ that is conjugate to $\underline{P}$, it holds that $\overline{P}(A) = \sum_{B \cap A \neq \emptyset} m(B)$ for every $A \subseteq \Omega$.

For the results in this section, it shall be interesting to work with the focal elements of the belief function:

Definition 7. [24] Given a belief function $\underline{P}$ with Möbius inverse m, a subset $B \subseteq \Omega$ is called a *focal element* when $m(B) > 0$. The union F of all the focal elements of $\underline{P}$ is called the *core* of $\underline{P}$.

We shall be particularly interested in those belief functions whose focal elements cover the possibility space Ω:

Definition 8. A belief function $\underline{P}$ with core F is called *full* when $F = \Omega$.

Since $\overline{P}(F^c) = \sum_{B \text{ focal}: B \cap F^c \neq \emptyset} m(B) = 0$, given a belief function that is not full, any set included in F^c will have zero upper probability. Equivalently, if $\underline{P}$ is a full belief function, any subset B of Ω has a positive upper probability.

Recall that for any conditioning event B, it holds that $\underline{E}(\cdot|B) = \underline{R}(\cdot|B)$ if $\underline{P}(B) > 0$ or $\overline{P}(B) = 0$. Hence, the natural and regular extensions will agree as soon as there is no conditioning event with zero lower probability and positive upper probability. This case is characterised by the following definition:

Definition 9. A belief function is called *non-atomic* if for every focal element B, it holds that $m(\{\omega\}) > 0$ for every $\omega \in B$.

The reason for this terminology is that given such a belief function there is no set B with $|B| \geq 2$ satisfying $\underline{P}(B) > 0$ and $\underline{P}(A) = 0$ for every $A \subsetneq B$. See [1, 19] for related concepts. Non-atomic and full belief functions can be characterised in the following way:

Proposition 6. *Let $\underline{P}$ be a belief function on $\mathcal{P}(\Omega)$.*

1. *$\underline{P}$ is non-atomic if and only if for any $B \subseteq \Omega$ either $\overline{P}(B) = 0$ or $\underline{P}(B) > 0$.*
2. *$\underline{P}$ is full if and only if for any $B \subseteq \Omega$, $\overline{P}(B) > 0$.*

3. $\underline{P}$ *is full and non-atomic if and only if* $\underline{P}(B) > 0$ *for every* $B \subseteq \Omega$.

When the conditioning event B has zero lower probability and positive upper probability the equality between the natural and the regular extensions is characterised by Proposition 5: we need that $\overline{P}(\{\omega\}) > 0$ for every $\omega \in B$; in the case of belief functions, this is equivalent to $B \subseteq F$, the core of the belief function. From this we deduce the following result:

Proposition 7. *Let* $\underline{P}$ *be a completely monotone lower prevision on* $\mathcal{L}(\Omega)$*, and let* μ *denote the belief function that is the restriction of* $\underline{P}$ *to events. Then,* $\underline{E}(\cdot|B) = \underline{R}(\cdot|B)$ *for every* $B \subseteq \Omega$ *if and only if* μ *is either full or non-atomic.*

This result allows to provide an example where the natural and the regular extensions do not coincide:

Example 2. Consider $\Omega = \{a, b, c, d\}$, and let $\underline{P}$ be the completely monotone lower prevision given by

$$\underline{P}(f) = \min\{f(b), f(c)\} \; \forall f \in \mathcal{L}(\Omega).$$

The restriction to events of $\underline{P}$ is the belief function associated to the basic probability assignment m where

$$m(\{b, c\}) = 1 \text{ and } m(C) = 0 \;\text{ for every } C \neq \{b, c\}.$$

Obviously, this belief function is not full. If we take $B = \{a, b\}$ and $A = \{b\}$, then any probability $P \geq \underline{P}$ satisfying $P(B) > 0$ must satisfy $P(\{b\}) > 0$, because $P(\{a\}) \leq \overline{P}(\{a\}) = 0$. But then P will satisfy $P(A|B) = 1$, and from this we deduce that

$$\underline{R}(A|B) = 1 > 0 = \underline{E}(A|B),$$

where the last equality holds because $\underline{P}(B) = 0$. Hence, the natural and regular extensions do not coincide. ♦

Moreover, for completely monotone lower previsions we can give an alternative expression of the regular extension to that in Proposition 4.

Proposition 8. *Let* $\underline{P}$ *be a completely monotone lower prevision, and let* F *be the core of its associated belief function. Then for any* $B \subseteq \Omega$ *such that* $\underline{P}(B) = 0 < \overline{P}(B)$*,*

$$\underline{R}(f|B) = \min_{\omega \in B \cap F} f(\omega) \; \forall f \in \mathcal{L}(\Omega).$$

Let us recall again that the condition $\underline{P}(B) = 0 < \overline{P}(B)$ we consider in this theorem implies that the belief function is not non-atomic.

From Proposition 7 we immediately derive the following theorem.

Theorem 2. *Let* $\underline{P}$ *be a completely monotone lower prevision on* $\mathcal{L}(\Omega)$ *and let* $\mathcal{B}$ *be a partition of* Ω*. If the restriction to events* μ *of* $\underline{P}$ *is either full or non-atomic, then* $\underline{E}(\cdot|\mathcal{B}) = \underline{R}(\cdot|\mathcal{B})$.

Note that the sufficient condition in this theorem is not necessary: it may be that μ is neither full nor non-atomic and $\mu(B) > 0$ for every B in the partition $\mathcal{B}$, and then $\underline{E}(\cdot|\mathcal{B}) = \underline{R}(\cdot|\mathcal{B})$.

4.1 Random Sets

One context where completely monotone lower previsions arise naturally is that of measurable multi-valued mappings, or random sets [11, 23].

Definition 10. Let $(X, \mathcal{A}, P)$ be a probability space, $(\Omega, \mathcal{P}(\Omega))$ a measurable space, where Ω is finite, and $\Gamma : X \to \mathcal{P}(\Omega)$ a non-empty multi-valued mapping. It is called a *random set* when it satisfies the following measurability condition:

$$\Gamma_*(A) := \{x \in X : \Gamma(x) \subseteq A\} \in \mathcal{A} \quad \forall A \subseteq \Omega.$$

Its associated *lower probability* $P_{*\Gamma} : \mathcal{P}(\Omega) \to [0, 1]$ is a belief function and is given by

$$P_{*\Gamma}(A) = P(\Gamma_*(A)) \quad \forall A \subseteq \Omega. \tag{6}$$

The focal elements of $P_{*\Gamma}$ are given by

$$\{A \subseteq \Omega : P(\Gamma^{-1}(A)) > 0\},$$

and its Möbius inverse is given by $m = P \circ \Gamma^{-1}$. The conjugate plausibility measure is denoted by P^*_Γ and it is called the upper probability of the random set Γ. It satisfies

$$P^*_\Gamma(A) = 1 - P_{*\Gamma}(A^c) = P(\{x : \Gamma(x) \cap A \neq \emptyset\}),$$

where the set $\{x : \Gamma(x) \cap A \neq \emptyset\}$ is the *upper inverse* of A by Γ, and is usually denoted by Γ^*. The Choquet integral with respect to $P_{*\Gamma}$ is a completely monotone lower prevision on $\mathcal{L}(\Omega)$, and it corresponds to the natural extension of $P_{*\Gamma}$ from $\mathcal{P}(\Omega)$ to the set of all gambles. If we want to update this completely monotone lower prevision, we can use the natural or the regular extensions, that, by Proposition 7, coincide if and only if $P_{*\Gamma}$ is either full or non-atomic. These properties can be easily characterised in terms of the images of Γ:

Proposition 9. *Let* $(X, \mathcal{A}, P)$ *be a probability space,* Ω *a finite set and* $\Gamma : X \to \mathcal{P}(\Omega)$ *a random set with associated lower probability* $P_{*\Gamma}$*. Let* F *denote the core of* $P_{*\Gamma}$*.*

1. $P_{*\Gamma}$ *is full* $\Leftrightarrow F = \Omega \Leftrightarrow P^*_\Gamma(B) > 0$ *for all* $B \subseteq X \Leftrightarrow P^*_\Gamma(\{\omega\}) > 0$ *for all* $\omega \in \Omega \Leftrightarrow P(\{x : \omega \in \Gamma(x)\}) > 0$ *for all* $\omega \in \Omega$.

2. $P_{*\Gamma}$ *is non-atomic* $\Leftrightarrow \forall \omega \in F, P(\Gamma^{-1}(\omega)) > 0$.

Moreover, $\underline{E}(\cdot|B) = \underline{R}(\cdot|B)$ *for all* $B \subseteq \Omega$ *if and only if* $P_{*\Gamma}$ *is either full or non-atomic.*

One interesting interpretation of random sets is the *epistemic* one, where they are seen as models for the imprecise knowledge of a random variable [16]. In that case, our information about this random variable is provided by the *measurable selections* of Γ: those measurable mappings $U : X \to \Omega$ such that $U(x) \in \Gamma(x)\ \forall x \in X$. We shall denote by $S(\Gamma)$ the set of measurable selections of Γ and by $P(\Gamma)$ the set of the probability measures they induce on $\mathcal{P}(\Omega)$. This set is included in the class $\mathcal{M}(P_{*\Gamma})$ of probabilities that dominate $P_{*\Gamma}$. Although both sets do not coincide in general, when Ω is finite it can be checked that:

Proposition 10. *[21, Theorem 1] Let* $\Gamma : X \to \mathcal{P}(\Omega)$ *be a random set, where* Ω *is finite. Then* $Ext(M(P_{*\Gamma})) \subseteq P(\Gamma)$ *and* $M(P_{*\Gamma}) = Conv(Ext(M(P_{*\Gamma})))$.

Moreover, from [11], $M(P_{*\Gamma})$ has a finite number of extreme points, that are related to the permutations of the final space.

The epistemic interpretation can be carried on towards the regular extension, in the following sense:

Proposition 11. *Let* $(X, \mathcal{A}, P)$ *be a probability space,* Ω *a finite set and* $\Gamma : X \to \mathcal{P}(\Omega)$ *a random set with associated lower probability* $P_{*\Gamma}$*. Consider* $B \subseteq \Omega$ *with* $P^*_\Gamma(B) > 0$*. Then, for every* $f \in \mathcal{L}(\Omega)$*,*

$$\underline{R}(f \mid B) = \min\{P_U(f \mid B) : U \in S(\Gamma), P_U(B) > 0\}.$$

To conclude this section, we use random sets to establish that, even if the conditional lower probability derived from a completely monotone lower prevision by Generalised Bayes Rule is a belief function [14, 15], when we move from events to gambles we do not necessarily obtain a completely monotone lower prevision.

Example 3. Consider the probability space $(X, \mathcal{P}(X), P)$, where $X = \{a, b, c, d, e\}$, and P is the probability measure determined by the equalities $P(a) = P(b) = 1/8$, and $P(c) = P(d) = P(e) = 1/4$. Let Γ be the multi-valued mapping $\Gamma : X \to \mathcal{P}(\{1, 2, 3, 4\})$ given by $\Gamma(a) = \{1\}, \Gamma(b) = \{2\}, \Gamma(c) = \{1, 4\}, \Gamma(d) = \{2, 4\}, \Gamma(e) = \{3, 4\}$.

Let $P_{*\Gamma}$ denote the lower probability induced by this random set. This is a belief function, and the lower prevision $\underline{P}$ on $\mathcal{L}(\{1, 2, 3, 4\})$ given by $\underline{P}(f) = (C)\int f dP_{*\Gamma}$ is a completely monotone lower prevision.

It follows from Eq. (6) that

$$P_{*\Gamma}(\{1, 2, 3\}) = P(\{a, b\}) = \frac{1}{4} > 0.$$

As a consequence, the natural and regular extensions coincide, and we deduce from Proposition 11 that

$$\underline{R}(f|\{1, 2, 3\}) = \min\{P_U(f|\{1, 2, 3\}) : U \in S(\Gamma)\}. \tag{7}$$

Let us consider the gamble f on $\{1, 2, 3, 4\}$ given by $f(\omega) = 4 - \omega$ for all $\omega \in \{1, 2, 3, 4\}$. Then since $f = 1\,\mathbb{I}_{1,2,3} + 1\,\mathbb{I}_{1,2} + 1\,\mathbb{I}_1$, its Choquet integral with respect to $\underline{R}(\cdot|\{1, 2, 3\})$ would be

$$1 + \underline{R}(\{1, 2\}|\{1, 2, 3\}) + \underline{R}(\{1\}|\{1, 2, 3\}).$$

We deduce from Eq. (7) that

$$\underline{R}(\{1\}|\{1, 2, 3\}) = \frac{1}{6} \text{ and } \underline{R}(\{1, 2\}|\{1, 2, 3\}) = \frac{1}{2};$$

as a consequence, $(C)\int f\ d\underline{R}(\cdot|\{1, 2, 3\}) = 5/3$.

On the other hand, the smallest value of $\{P_U(f|\{1, 2, 3\}) : U \in S(\Gamma)\}$ is given by $7/4 > 5/3$. This means that $\underline{R}(f|\{1, 2, 3\}) > (C)\int f d\underline{R}(\cdot|\{1, 2, 3\})$.

But it has been established in [7, 26] that if we have a 2-monotone lower probability on all events (as is the case for $\underline{R}(\cdot|\{1, 2, 3\})$), the *only* 2-monotone extension to all gambles is the Choquet integral. This means that the conditional lower prevision $\underline{R}(\cdot|\{1, 2, 3\})$ is not 2-monotone on $\mathcal{L}(\{1, 2, 3\})$. ♦

5 Coherent updating of minimum-preserving previsions

We consider now the particular case of completely monotone lower previsions that are minimum-preserving, i.e., lower previsions $\underline{P}$ such that

$$\underline{P}(f \wedge g) = \min\{\underline{P}(f), \underline{P}(g)\}$$

for every pair of gambles f, g on Ω. They correspond to the Choquet integral with respect to their restriction to events, which is a necessity measure N. Their conjugate upper previsions $\overline{P}$ are the Choquet integral with respect to the possibility measure Π that is determined by N using duality, and are maximum-preserving.

From Proposition 7, we deduce the following:

Corollary 1. *Let* $\underline{P}$ *be a minimum-preserving coherent lower prevision. Then* $\underline{E}(\cdot|B) = \underline{P}(\cdot|B)$ *for all* $B \subseteq \Omega$ *if and only if either of the following conditions holds:*

(i) $\overline{P}(\{\omega\}) > 0$ *for all* $\omega \in \Omega$*.*

(ii) $\underline{P}(\{\omega\}) = 1$ *for some* $\omega \in \Omega$.

The result in Corollary 1 can be simplified further taking into account that de Cooman and Aeyels proved in [5] (see also [6]) that a coherent *upper* prevision $\overline{P}$ on $\mathcal{L}(\Omega)$ is maximum-preserving if and only if its restriction to events is a 0–1-valued possibility measure. Then, if we define $F := \{\omega : \overline{P}(\{\omega\}) = 1\}$, it turns out that F is the only focal element of the possibility measure $\overline{P}$, and $m(F) = 1$. Hence, $\overline{P}$ is the vacuous lower prevision on F, that is,

$$\underline{P}(f) = \min_{\omega \in F} f(\omega) \quad \forall f \in \mathcal{L}(\Omega).$$

Now, given a conditioning event $B \subseteq F$, there are a number of possibilities:

- $B \subseteq F^c$. Then $\overline{P}(B) = 0$ and both the natural and regular extensions are vacuous.
- $B \cap F \neq \emptyset \neq B \cap F^c$. Then $\underline{P}(B) = 0 < 1 = \overline{P}(B)$, whence $\underline{E}(\cdot|B)$ is vacuous on B and $\underline{R}(\cdot|B)$ is vacuous on $B \cap F$. Hence, in that case the natural and regular extensions do not coincide.
- $B \subseteq F$. Then both $\underline{E}(\cdot|B)$ and $\underline{R}(\cdot|B)$ are vacuous on B.

Note that in this case $\underline{P}$ is only non-atomic when F is a singleton (i.e., when $\underline{P}$ corresponds to the expectation operator with respect to a degenerate probability measure), and $\underline{P}$ is full if and only if $F = \Omega$, meaning that $\underline{P}$ corresponds to the vacuous model. Hence, we only have the equality between the natural and the regular extensions for all $B \subseteq \Omega$ in these two extreme cases.

We summarise the coherent updating of a minimum-preserving lower prevision in the following theorem.

Theorem 3. *Let $\underline{P}$ be a minimum-preserving lower prevision on $\mathcal{L}(\Omega)$, and consider a partition $\mathcal{B}$ of Ω. Consider $F \subseteq \Omega$ such that $\underline{P}(f) = \min_{\omega \in F} f(\omega) \forall f \in \mathcal{L}(\Omega)$. Given $B \in \mathcal{B}$ and $f \in \mathcal{L}(\Omega)$,*

1. $$\underline{E}(f|B) = \begin{cases} \min_{\omega \in B} f(\omega) & \text{if } F \not\subseteq B \\ \min_{\omega \in F} f(\omega) & \text{if } F \subseteq B. \end{cases}$$
2. $$\underline{R}(f|B) = \begin{cases} \min_{\omega \in B \cap F} f(\omega) & \text{if } B \cap F \neq \emptyset \\ \min_{\omega \in B} f(\omega) & \text{if } B \cap F = \emptyset. \end{cases}$$
3. $\underline{E}(f|B) = \underline{R}(f|B)$ *if and only if either* $B \cap F = \emptyset$ *or* $B \cap F^c = \emptyset$.
4. *A separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ is coherent with $\underline{P}$ if and only if*
$$\min_{\omega \in B} f(\omega) \leq \underline{P}(f|B) \leq \min_{\omega \in B \cap F} f(\omega)$$
for every $f \in \mathcal{L}(\Omega), B \in \mathcal{B}$ *s.t* $B \cap F \neq \emptyset$.

From Theorem 3, the bounds determined by natural and regular extension are both minimum-preserving, and as a consequence they correspond to the Choquet integral of their respective restrictions to events. However, it is easy to see that not every separately coherent conditional lower prevision between them is minimum-preserving.

5.1 Comparison with the updating of possibility measures

The results in this paper allow us to show one interesting phenomenon: that, even if a minimum-preserving lower prevision $\underline{P}$ is the natural extension of its restriction to events N, the coherence of N with a conditional lower probability $N(\cdot|\mathcal{B})$ is not equivalent to the coherence of the lower previsions $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$ that each of them determines by natural extension. This is the reason behind the apparent contradiction with the results in [30]: it is showed there that Dempster's rule is a coherent updating rule for updating a possibility measure, even if it can be more informative than the conditional possibility we obtain by regular extension.

To make this clearer, let us study the results in [30] in more detail. The authors consider two finite sets $\mathcal{X}$ and $\mathcal{Y}$, and let $\Omega = \mathcal{X} \times \mathcal{Y}$. They take a possibility measure $\Pi(\cdot,\cdot)$ on $\mathcal{P}(\Omega)$ and look for the smallest and greatest conditional possibility measures $\Pi(\cdot|Y)$ that satisfy coherence with Π. Note that, since we are dealing with upper previsions now, it follows from conjugacy and Proposition 2 that a conditional upper prevision $\overline{P}(\cdot|\mathcal{B})$ is coherent with $\overline{P}$ if and only if $\overline{P}(f|B) \in [\overline{R}(f|B), \overline{E}(f|B)]$ for every gamble f and every $B \subseteq \Omega$ s.t. $\overline{P}(B) > 0$, where $\overline{R}(\cdot|B)$ and $\overline{E}(\cdot|B)$ are the conjugate upper previsions of the regular and natural extensions, respectively.

In [30], the focus is on conditional upper *probabilities* instead of previsions, and in particular on those conditional possibility measures $\Pi(\cdot|Y)$ that satisfy coherence with the unconditional possibility measure Π. They prove in [30, Theorem 4] that the greatest such conditional possibility measure is given by natural extension, while the smallest such conditional possibility measure is determined by Dempster's rule, which produces the possibility measure associated to the following possibility distribution:

$$\pi_{DE}(x|y) = \begin{cases} \frac{\pi(x,y)}{\pi(y)} & \text{if } \pi(y) > 0 \\ 1 & \text{if } \pi(y) = 0. \end{cases}$$

Then in [30], it is advocated to use the harmonic mean between Dempster's rule and natural extension as an informative updating rule for updating a possibility measure Π. This harmonic mean determines the possibility measure defined by the possibility distribution $\pi_{HM}(x|y)$ given by

$$\begin{cases} \frac{2\pi(x,y)}{\pi(x,y)+\pi(y)+1-\max\{\pi(x,y),\Pi(\{y\}^c)} & \text{if } \pi(y) > 0 \\ 1 & \text{if } \pi(y) = 0. \end{cases}$$

However, this rule may be dominated by the regular extension, that produces the conditional possibility measure $\pi_{RE}(x|y)$ given by

$$\begin{cases} \frac{\pi(x,y)}{\pi(x,y)+1-\max\{\pi(x,y),\Pi(\{y\}^c)} & \text{if } \pi(y^c) < 1 \\ 0 & \text{if } \Pi(\{y\}^c) = 1, \pi(y) > \pi(x,y) = 0 \\ 1 & \text{otherwise,} \end{cases}$$

and as a consequence it is not a valid updating rule if we are working with upper previsions instead of upper probabilities. Consider the following example:

Example 4. Consider $\mathcal{X} = \{x_1, x_2\}, \mathcal{Y} = \{y_1, y_2\}$ and let Π be the possibility measure associated to the possibility distribution $\pi(x_1, y_1) = 0.3, \pi(x_1, y_2) = 1, \pi(x_2, y_1) = 0.5$ and $\pi(x_2, y_2) = 0.2$. Then it can be checked that the conditional possibility measure determined by the harmonic mean satisfies $\pi_{HM}(x_2|y_2) = 0.235$, whereas both the natural and the regular extensions produce $\pi_{NE}(x_2|y_2) = \pi_{RE}(x_2|y_2) = 0.285$. Thus, the conditional possibility measure determined by the harmonic mean is dominated by the one produced by regular extension, and as a consequence the conditional upper prevision determined by means of the Choquet integral with respect to $\Pi_{HM}(X|Y)$ is not coherent with the unconditional upper prevision associated to Π. ♦

6 Conclusions

In this work we have considered the problem of updating a coherent lower prevision into a conditional one, while preserving the property of coherence. This problem has a simple solution when the conditioning event has a positive lower probability, as showed by Walley in [27]: it suffices to apply Generalised Bayes Rule. However, when the conditioning event has zero lower probability and strictly positive upper probability, there may be an infinite number of coherent updated models. In that case, it becomes necessary to determine a rule to elicit the appropriate one for the problem at hand. Here, we have studied in which cases we can skip this situation, because the procedures of natural and regular extension give rise to the same updated model. We have considered the particular case when our unconditional model satisfies the property of 2-monotonicity, which guarantees that the lower prevision is the Choquet integral of the coherent lower probability that is its restriction to events, and we have obtained necessary and sufficient conditions for the equality between the natural and regular extensions. As particular cases, we have also considered the updating problem for completely monotone lower previsions, random sets and minimum-preserving previsions.

It is interesting to remark that the conditional lower probabilities determined by the natural and regular extension preserve the property of n-monotonicity from the unconditional model; in fact, when the conditioning event has zero lower probability and positive upper probability, they are moreover minimum-preserving. However, the conditional lower previsions they determine are not necessarily 2-monotone, even if we start from a completely monotone coherent lower prevision, as we have showed in Example 3. On the other hand, the properties of the natural and the regular extension are not shared in general by all the conditional models that are coherent with the unconditional one.

Finally, let us stress once again that, even if the property of 2-monotonicity means that the lower prevision is uniquely determined by its lower probability, the problem of coherently updating 2-monotone lower probabilities is not equivalent to that of updating 2-monotone lower previsions; this can be seen from the results in Section 5.1.

With respect to the open problems arising from this work, perhaps the most important one would be the extension of our results to infinite spaces. Although some work in this direction was already carried out in [20], we expect the problem to be much more difficult; one of the reasons is that the coherence condition between the unconditional and conditional lower previsions must take into account the property of conglomerability. See [27, Chapter 6] and [22] for more details. Another interesting line of research may be the extension of our work to the updating by means of several partitions. In that case, we should distinguish between the notions of weak and strong coherence studied by Walley in [27, Chapter 7].

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References

[1] R. Aumann and L. S. Shapley. *Values of non-atomic games*. Princeton University Press, 1974.

[2] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953–1954.

[3] G. Coletti and R. Scozzafava. *Probabilistic logic in a coherent setting*. Kluwer, 2002.

[4] L. M. de Campos, M. T. Lamata, and S. Moral. The concept of conditional fuzzy measures. *International Journal of Intelligent Systems*, 5:237–246, 1990.

[5] G. de Cooman and D. Aeyels. Supremum preserving upper probabilities. *Information Sciences*, 118:173–212, 1999.

[6] G. de Cooman and E. Miranda. Lower previsions induced by filter maps. Submitted for publication, 2012.

[7] G. de Cooman, M. C. M. Troffaes, and E. Miranda. n-Monotone exact functionals. *Journal of Mathematical Analysis and Applications*, 347:143–156, 2008.

[8] G. de Cooman and M. Zaffalon. Updating beliefs with incomplete observations. *Artificial Intelligence*, 159(1-2):75–125, 2004.

[9] B. de Finetti. *Teoria delle Probabilità*. Einaudi, Turin, 1970.

[10] B. de Finetti. *Theory of Probability: A Critical Introductory Treatment*, volume 1. John Wiley & Sons, Chichester, 1974. English translation of [9].

[11] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, 38:325–339, 1967.

[12] D. Denneberg. *Non-Additive Measure and Integral*. Kluwer Academic, Dordrecht, 1994.

[13] L. E. Dubins. Finitely additive conditional probabilities, conglomerability and disintegrations. *The Annals of Probability*, 3:88–99, 1975.

[14] R. Fagin and J. Y. Halpern. A new approach to updating beliefs. In P. P. Bonissone, M. Henrion, L. N. Kanal, and J. F. Lemmer, editors, *Uncertainty in Artificial Intelligence*, volume 6, pages 347–374. North-Holland, Amsterdam, 1991.

[15] J.-Y. Jaffray. Bayesian updating and belief functions. *IEEE Transactions on Systems, Man and Cybernetics*, 22:1144–1152, 1992.

[16] R. Kruse and K. D. Meyer. *Statistics with vague data*. D. Reidel Publishing Company, Dordrecht, 1987.

[17] V. P. Kuznetsov. *Interval Statistical Methods*. Radio i Svyaz Publ., 1991. (in Russian).

[18] I. Levi. *The enterprise of knowledge*. MIT Press, Cambridge, 1980.

[19] M. Marinacci and L. Montrucchio. Introduction to the mathematics of ambiguity. In I. Gilboa, editor, *Uncertainty in economic theory*. Routledge, New York, 2004.

[20] E. Miranda. Updating coherent lower previsions on finite spaces. *Fuzzy Sets and Systems*, 160(9):1286–1307, 2009.

[21] E. Miranda, I. Couso, and P. Gil. Upper probabilities and selectors of random sets. In P. Grzegorzewski, O. Hryniewicz, and M. A. Gil, editors, *Soft methods in probability, statistics and data analysis*, pages 126–133. Physica-Verlag, Heidelberg, 2002.

[22] E. Miranda, M. Zaffalon, and G. de Cooman. Conglomerable natural extension. *International Journal of Approximate Reasoning*, 53(8):1200–1227, 2012.

[23] H. T. Nguyen. *An introduction to random sets*. Chapman and Hall, 2006.

[24] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, NJ, 1976.

[25] C. Sundberg and C. Wagner. Generalized finite differences and Bayesian conditioning of Choquet capacities. *Advances in Applied Mathematics*, 13(3):262–272, 1992.

[26] P. Walley. Coherent lower (and upper) probabilities. Technical Report Statistics Research Report 22, University of Warwick, Coventry, 1981.

[27] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

[28] P. Walley. Inferences from multinomial data: learning about a bag of marbles. *Journal of the Royal Statistical Society, Series B*, 58:3–57, 1996. With discussion.

[29] P. Walley. Measures of uncertainty in expert systems. *Artificial Intelligence*, 83(1):1–58, 1996.

[30] P. Walley and G. de Cooman. Coherence of rules for defining conditional possibility. *International Journal of Approximate Reasoning*, 21:63–107, 1999.

[31] P. Williams. On an new theory of epistemic probability. *British Journal for the Philosophy of Science*, 29:375–387, 1978.