Conditional non–additive measures and fuzzy sets

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Abstract

Consistency of partial assessments with different frameworks (probability, possibility, plausibility) is studied. We are interested in inferential processes like the Bayesian one, with particular attention when a part of the information is expressed in natural language and can be modeled by a possibilistic or a plausibilistic likelihood.

Keywords. Natural extension, conditional Plausibilities, T-conditional possibilities, generalized Bayesian inference, fuzzy sets.

1 Introduction

Fuzzy set theory, introduced by Zadeh [42], has become very popular and it provides a formalization of some concepts expressed by means of natural language. Different interpretations of fuzzy sets have been given [35, 26, 38] in terms of (conditional) probabilities, we refer to that given in [9, 10, 8], where the membership function of a fuzzy subset is interpreted in terms of a coherent conditional probability assessment. This interpretation, as shown in [5, 14, 13], is particularly useful when fuzzy and statistical information is simultaneously available.

Nevertheless sometimes the statistical information is related to a family of events different from that of interest and in which the fuzzy information is available (as a particular case we can have two partitions such that the elements of one are finite conjunctions of the element of the others): by extending the probabilistic assessment a la de Finetti [20, 41] we obtain a family of probabilities, whose upper envelope, which is in general only an upper probability, could be a plausibility [23, 31, 40, 11] or a possibility [28, 15, 22].

In this paper we consider the above problems by focusing mainly on plausibility and possibility measures, for which many proposals of conditioning are present. We adopt the definition of T-conditional possibility, with T any t-norm (introduced in [3] for minimum and generalized in [17] for any t-norm): this class of conditional measures includes as a particular case the conditional possibilities obtained by using the Dubois and Prade rule based on minimum specificity principle [27]. For conditional plausibility we adopt a definition generalizing Dempster rule, introduced in [6, 36], also if, as it is well known, it cannot be obtained as the lower envelope of a class of conditional probabilities. Nevertheless it assures a "weak disintegration rule" and admits as particular case T-conditional possibility, with T the usual product.

In the first part (Section 2 and 3) of the paper, in order to consider a generalized Bayesian inferential procedure, by using the concept of coherence (that is the consistency of a partial assessment with a conditional possibility or plausibility), we study the properties of likelihood functions, both as point and set functions, in the different frameworks. Moreover, we study the coherence of a likelihood with a plausibility (or possibility) measure having the role of "a prior".

In Section 4 we give an interpretation of the membership of fuzzy sets as a possibilistic or a plausibilistic likelihood function and we study which properties of fuzzy set theory are maintained. In both cases the semantic of the interpretation seem to be very similar: if φ is a property, related to a variable X, the meaning associated to the membership $\mu_\varphi(x)$ on x consists into the possibility [plausibility] that You claim that X is φ under the hypothesis that X assumes the value x. We show that from a syntactical point of view many differences and common features can occur. About the specific feature the most relevant is that the membership $\mu_{\varphi\vee\psi}$ of the union of two fuzzy sets, with memberships μ_φ and μ_ψ, is not linked to $\mu_{\varphi\wedge\psi}$ by the Frank equation ([30]), as in probability theory. On the contrary, in the case of possibilistic setting $\mu_{\varphi\vee\psi}$ is univocally determined by μ_φ and μ_ψ independently of $\mu_{\varphi\wedge\psi}$. While in the case of plausibilistic framework it is not univocally determined, but $\mu_{\varphi\vee\psi}(x)$ must be between $\max\{\mu_\varphi(x), \mu_\psi(x)\}$ and $\min\{\mu_\varphi(x) + \mu_\psi(x)\} - \mu_{\varphi\wedge\psi}(x), 1\}$.

In this interpretation the fuzzy membership μ_φ coincides with a likelihood and the fuzzy event E_φ is the Boolean event "You claim that X is φ"; moreover for the measure of uncertainty of E_φ when the prior on X is a plausibility we get an upper bound, while when the prior is a possibility we give an analytic formula depending on the chosen t-norm.

2 Conditional measures

Usually in literature a conditional measure is presented as a derived notion of the unconditional one, by introducing a law involving the joint measure and its marginal. Nevertheless, this could be restrictive, since for some pair of events the solution of the equation (the conditional measure) can either not exists or to be not unique. So, in analogy with conditional probability [21], it is preferable to define conditional measures in an axiomatic way, directly as a function defined on a suitable set of conditional events. We recall here the notion of T-conditional possibility (with T any t-norm)[3, 17]

Definition 1. *Let T be any t-norm. Given a Boolean algebra $\mathcal{B}$ and an additive set (closed under finite disjunctions) $\mathcal{H}$ with $\mathcal{H} \subseteq \mathcal{B}^0 = (\mathcal{B} \setminus \{\emptyset\})$, a function $\Pi : \mathcal{B} \times \mathcal{H} \to [0,1]$ is a T-conditional possibility if it satisfies the following properties:*

(i) $\Pi(E|H) = \Pi(E \wedge H|H)$, *for every* $E \in \mathcal{B}$ *and* $H \in \mathcal{H}$*;*

(ii) $\Pi(\cdot|H)$ *is a (finitely maxitive) possibility on* $\mathcal{B}$*, for any* $H \in \mathcal{H}$*;*

(iii) $\Pi(E \wedge F|H) = T(\Pi(E|H), \Pi(F|E \wedge H))$*, for any* $H, E \wedge H \in \mathcal{H}$ *and* $E, F \in \mathcal{B}$*.*

Condition *(ii)* of previous definition requires that $\Pi(\Omega|H) = 1$, $\Pi(\emptyset|H) = 0$ and for every $H \in \mathcal{H}$, $\Pi(\bigvee_{i=1,...,n} A_i|H) = \max_{i=1,...,n} \Pi(A_i|H)$, for every $A_1, ..., A_n \in \mathcal{B}$ [37]. Moreover from *(i)* and *(ii)* $\Pi(H|H) = 1$ for every $H \in \mathcal{H}$.

Actually, conditional possibility (according to Definition 1) cannot be in general induced by a unique possibility, but by a class of possibilities (for more details, see [17]). Nevertheless, by using some principle, conditional possibility could be defined by means of a unique possibility measure. Obviously some principles can give rise to assessments inconsistent with axioms *(i) – (iii)*, see [16, 17].

Taken the minimum t-norm, by considering the minimum specificity principle the following notion of conditioning [27] arises (in the following called DP-conditional possibility, where DP stands for Dubois and Prade):

for any $E|H$ in $\mathcal{B}\times\mathcal{H}^0$, $\Pi(E|H) = 1$, when $\Pi(E\wedge H) = \Pi(H)$ and $E \wedge H \neq \emptyset$, $\Pi(E|H) = \Pi(E\wedge H)$ otherwise.

It is easy to see that a DP-conditional possibility is a conditional possibility in the sense of Definition 1. More generally, for a continuous t-norm, the T-conditional possibility $\Pi(E|H)$ can be seen as the residuum $\to_T$ of the t-norm T

$x \to_T y = \sup\{z \in [0,1] : T(x,z) = y\}$

that means $\Pi(H) \to_T \Pi(E \wedge H)$ whenever $E \wedge H \neq \emptyset$ (see [19]). In [2] a link between these kinds of conditioning and Jeffrey's rule is studied, while in [25] connections between conditioning in possibility and belief function context are studied.

In [17] we proved that if T is a continuous t-norm, a conditional possibility can be extended on any other set $\mathcal{B}' \times \mathcal{H}'$ with $\mathcal{B}'$ a Boolean algebra and $\mathcal{H}'$ an additive set ($\mathcal{H}' \subseteq \mathcal{B}^0$) with $\mathcal{B}\times\mathcal{H} \subset \mathcal{B}'\times\mathcal{H}'$. Moreover, for any $E|H$ in $\mathcal{B}' \times \mathcal{H}' \setminus \mathcal{B} \times \mathcal{H}$ the admissible values lay on a closed interval.

Analogously, conditional plausibility can be defined axiomatically as follows (see [6, 11]):

Definition 2. *Let $\mathcal{B}$ be a Boolean algebra and $\mathcal{H} \subseteq \mathcal{B}^0$ an additive set. A function Pl defined on $\mathcal{C} = \mathcal{B} \times \mathcal{H}$ is a conditional plausibility if it satisfies the following conditions*

i) $Pl(E|H) = Pl(E \wedge H|H)$*;*

ii) $Pl(\cdot|H)$ *is a plausibility function* $\forall H \in \mathcal{H}$*;*

iii) For every $E \in \mathcal{B}$ *and* $H, K \in \mathcal{H}$

$$Pl(E \wedge H|K) = Pl(E|H \wedge K) \cdot Pl(H|K).$$

Moreover, given a conditional plausibility, a conditional belief function $Bel(\cdot|\cdot)$ is defined by duality as follows: for every event $E|H \in \mathcal{C}$

$$Bel(E|H) = 1 - Pl(E^c|H).$$

Condition *i)* and *ii)* requires that $Pl(\Omega|H) = Pl(H|H) = 1$ and $Pl(\emptyset|H) = 0$ and moreover, for any n, $Pl(\cdot|H)$ is n-alternating [23]:

$$Pl(A|H) \leq \sum (-1)^{|I|+1} Pl(\wedge_{i\in I} A_i|H) \qquad (1)$$

for any $A_1, ..., A_n, A \in A$ with $A = \vee_{i=1}^n A_i$. Then, $Bel(\cdot|H)$ is n-monotone for any n.

This axiomatization extends the Dempster's rule, i.e.

$$Bel(F|H) = 1 - \frac{Pl(F^c \wedge H)}{Pl(H)},$$

for all conditioning events H such that $Pl(H) > 0$. When all the conditioning events have positive plausibility, i.e. $Pl(H|H^0) > 0$ for any $H \in \mathcal{H}$ (with $H^0 = \vee_{H\in\mathcal{H}} H$), the above notions of conditional plausibility and conditional belief coincide with that given in [24]. In fact, if $Pl(H) > 0$ it follows

$$Bel(F|H) = \frac{Bel(F \vee H^c) - Bel(H^c)}{Pl(H)}. \qquad (2)$$

An easy consequence of Definition 2 is a weak form of disintegration formula for the plausibility of an event $E|H$ with respect to a partition $H_1, ..., H_N$ of H

$$Pl(E|H) \leq \sum_{k=1}^{N} Pl(H_k|H)Pl(E|H_k) \qquad (3)$$

Taking into the following definition of conditioning (see [29, 33, 40, 41]):

$$Pl(F|H) = \frac{Pl(F \wedge H)}{Pl(F \wedge H) + Bel(F^c \wedge H)} \qquad (4)$$

the obtained conditional plausibility Pl does not satisfy axiom *iii)* of Definition 2. Therefore conditional plausibilities given trough equation (4) does not satisfy equation (3).

Note that for T equal to the usual product every T-conditional possibility is a conditional plausibility.

In the next result we show that every conditional plausibility on $\mathcal{B} \times \mathcal{H}$ can be extended (not uniquely) to a full conditional plausibility on $\mathcal{B}$ (i.e., a conditional plausibility on $\mathcal{B} \times \mathcal{B}^0$).

Theorem 1. *Let $\mathcal{B}$ be a finite algebra. If Pl on $\mathcal{B} \times \mathcal{H} \to [0,1]$ is a conditional plausibility, then there exists a conditional plausibility $Pl' : \mathcal{B} \times \mathcal{B}^0 \to [0,1]$ such that $Pl'_{|\mathcal{B}\times\mathcal{H}} = Pl$.*

Proof. Denote $H_0^0 = \bigvee_{H\in\mathcal{H}} H$. If H_0^0 coincides with the certain event Ω, $Pl(\cdot|\Omega)$ defines univocally $Pl'(E|H)$ for $Pl(H|\Omega) > 0$. Let $\mathcal{H}_0^1 = \{H \in \mathcal{B}^0 : Pl(H|\Omega) = 0\}$, $H_0^1 = \bigvee_{\mathcal{H}_0^1} H$ belongs to $\mathcal{B}^0$ and $Pl'(H_0^1|\Omega) = 0$ since $Pl(H_0^1|\Omega) \leq \sum_{H\in\mathcal{H}_0^1} Pl(H|\Omega)$. If $H_0^1 \in \mathcal{H}$ again for $Pl(H|H_0^1) > 0$ $Pl'(\cdot|H)$ is univocally defined, so proceed as before.

While for $H_0^1 \notin \mathcal{H}$ check whether the set $\mathcal{K} = \{H \in \mathcal{H} : Pl(H|H_0^1)\}$ is not empty. If it is not empty, consider the event $K_1 = \bigvee_{H\in\mathcal{K}} H$ in $\mathcal{H}$ and $K_1 \subseteq H_0^1$. Define $Pl'(E|H_0^1) = Pl(E|K_1)$ for any $E \in \mathcal{B}$. Note that $Pl'(K|H_0^1) = 1$, $Pl'(K^c|H_0^1) = 0$ and $Pl'(\cdot|H_0^1)$ is a plausibility since $Pl(\cdot|K_1)$ is. Otherwise if $\mathcal{K}$ is empty define $Pl'(E|H_o^1) = 1$ for any $E \in \mathcal{B}$ such that $E \wedge H_0^1 \neq \emptyset$. It is easy to check that even in this case $Pl'(\cdot|H_0^1)$ is a plausibility.

Now, define $\mathcal{H}_0^2 = \{H \in \mathcal{B}^0 : Pl(H|H_0^1) = 0\}$ and proceed as before.

It is easy to check that Pl' satisfies the axioms *iii)* of Definition 2 and so it is a conditional plausibility. □

Now we show that every full conditional plausibility on $\mathcal{B}$ can be extended as a full conditional plausibility on every finite superalgebra $\mathcal{B}' \supseteq \mathcal{B}$.

Theorem 2. *Let $\mathcal{B}$ be a finite algebra and $\mathcal{B}' \supseteq \mathcal{B}$ a finite superalgebra. If $Pl : \mathcal{B} \times \mathcal{B}^0 \to [0,1]$ is a full conditional plausibility, then there exists a full conditional plausibility $Pl' : \mathcal{B}' \times \mathcal{B}'^0 \to [0,1]$ such that $Pl'_{|\mathcal{B}\times\mathcal{B}^0} = Pl$.*

Proof. For any $A' \in \mathcal{B}'$ consider the smallest event $A \in \mathcal{B}$ containing A', $A = \vee_{C\in\mathcal{B}:C\wedge A'\neq\emptyset} C$ and define $Pl'(A') = Pl(A)$.
Since for any $A', B' \in \mathcal{B}'$, $Pl(A \wedge B) = Pl'(A' \wedge B')$ the function Pl' is a plausibility and induces a full conditional plausibility on $\mathcal{B}'$. By construction for any $A|B \in \mathcal{B} \times \mathcal{B}^0$ it holds $Pl'(A|B) = Pl(A|B)$. □

Note that the full conditional plausibility on $\mathcal{B}'$ extending the given conditional plausibility is not unique, that one given in the proof of Theorem 2 is just an example.

2.1 Coherent conditional plausibility

Analogously to probability theory, it is possible to introduce a notion of coherence in the context of plausibility functions, as done for conditional probabilities [21] and also for T-conditional possibilities [17].

Definition 3. *A function (or assessment) $\gamma : \mathcal{C} \to [0,1]$, on a set of conditional events $\mathcal{C}$, is a coherent conditional plausibility (T-conditional possibility) iff there exists a full conditional plausibility Pl (full T-conditional possibility Π) on an algebra $\mathcal{B}$ such that $\mathcal{C} \subseteq \mathcal{B} \times \mathcal{B}^0$ and the restriction of Pl (Π) on $\mathcal{C}$ coincides with γ.*

For a characterization of (coherent) conditional possibility, with T-continuous t-norm, see [17, 1]. Theorem 3 characterizes (coherent) conditional plausibility functions in terms of a class of plausibilities.

Theorem 3. *Let $\mathcal{F} = \{E_1|F_1, E_2|F_2, \ldots, E_m|F_m\}$ and denote by $\mathcal{B}$ the algebra generated by $\{E_1, \ldots, E_m, F_1, \ldots, F_m\}$, $H_0^0 = \vee_{j=1}^m F_j$. For $Pl : \mathcal{F} \to [0,1]$ the following statements are equivalent:*

(a) Pl is a coherent conditional plausibility;

(b) there exists a class $\mathcal{P} = \{Pl_\alpha\}$ of plausibility functions such that $Pl_\alpha(H_0^\alpha) = 1$ and $H_0^\alpha \subset H_0^\beta$

for all $\beta < \alpha$, where H_0^α is the greatest (with respect to the inclusion) element of $\mathcal{K}$ for which $Pl_{(\alpha-1)}(H_0^\alpha) = 0$.

Moreover, for every $E_i|F_i$, there exists a unique index α such that $Pl_\beta(F_i) = 0$ for all $\alpha > \beta$, $Pl_\alpha(F_i) > 0$ and

$$Pl(E_i|F_i) = \frac{Pl_\alpha(E_i \wedge F_i)}{Pl_\alpha(F_i)}, \tag{5}$$

(c) all the following systems (S^α), with $\alpha = 0, 1, 2, ..., k \leq n$, admit a solution $\mathbf{X}^\alpha = (\mathbf{x}_1^\alpha, ..., \mathbf{x}_{j_\alpha}^\alpha)$ with $\mathbf{x}_j^\alpha = m_\alpha(H_j)$ $(j = 1, ..., j_\alpha)$:

$$(S^\alpha) = \begin{cases} \sum\limits_{H_k \wedge F_i \neq \emptyset} x_k^\alpha \cdot Pl(E_i|F_i) = \sum\limits_{H_k \wedge E_i \wedge F_i \neq \emptyset} x_k^\alpha, \ \forall F_i \subseteq H_0^\alpha \\ \sum\limits_{H_k \in H_0^\alpha} x_k^\alpha = 1 \\ x_k^\alpha \geq 0, \qquad \forall H_k \subseteq H_0^\alpha \end{cases}$$

where H_0^α is the greatest element of $\mathcal{K}$ such that $\sum\limits_{H_i \wedge H_0^\alpha \neq \emptyset} m_{(\alpha-1)}(H_i) = 0$.

In particular, conditions *(b)* and *(c)* stress that this conditional measure can be written in terms of a suitable class of basic assignments, instead of just one as in the classical case, where all the conditioning events have positive plausibility.

Note that every class $\mathcal{P}$ (condition *(b)* of Theorem 3) is said to be agreeing with conditional plausibility Pl. Whenever there are events in $\mathcal{K}$ with zero plausibility the class of unconditional plausibilities contains more than one element and we can say that Pl_1 gives a refinement of those events judged with zero plausibility under Pl_0.

For an example showing the construction of the class $\mathcal{P}$ characterizing (in the sense of the above result) a conditional plausibility see [36].

3 Likelihood functions

This section is devoted to a comparative analysis of likelihood functions under different frameworks: probability, possibility, plausibility.

Given an event E and a partition $\mathcal{L}$, a likelihood function is an assessment on $\{E|H_i : H_i \in \mathcal{L}\}$ (that is a function $f : \{E\} \times \mathcal{L} \to [0, 1]$) satisfying only the following trivial condition:

(L1) for every H_i such that $E \wedge H_i = \emptyset$ one has $f(E|H_i) = 0$ and for every H_i such that $H_i \subseteq E$ one has $f(E|H_i) = 1$

Theorem 4. *Let $\mathcal{L} = \{H_1, \ldots, H_n\}$ be a finite partition of Ω and E an event. For every likelihood function f on $\{E\} \times \mathcal{L}$ the following statements hold:*

a) f is a coherent conditional probability;

b) f is a coherent T-conditional possibility (for every continuous t-norm T);

c) f is a coherent conditional plausibility.

Proof. Condition *a)* and *b)* have been proved in [10] and [7], respectively.

Condition *c)* derives from *a)* and the fact that any coherent conditional probability is a coherent conditional plausibility (or equivalently from condition *b)* and the fact that any coherent T-conditional possibility, with T the usual product, is a coherent conditional plausibility). □

Theorem 5. *Let $\mathcal{L} = \{H_1, \ldots, H_n\}$ be a finite partition of Ω and E an event. If the only coherent conditional plausibility (possibility) f takes values in $\{0, 1\}$, then it is $H_i \wedge E = \emptyset$ for every H_i such that $f(E|H_i) = 0$ and it is $H_i \subseteq E$ for every H_i such that $f(E|H_i) = 1$.*

Proof. It follows directly from Theorem 3 and the characterization theorem for T-conditional possibilities [17]. □

The above results put in evidence that (in all contexts) no significant property characterizes likelihood as point function (i.e. an assessment on a partition).

This implies that since two likelihoods

$$f_i : \{E_i\} \times \mathcal{L}_i \to [0, 1]$$

$(i = 1, 2)$, related to events logically independent E_i are coherent with a conditional probability, then they should be coherent also with a conditional plausibility.

It is easy to show that $\{f_1, f_2\}$ are coherent also with a T-conditional possibility.

3.1 Likelihood and prior

The aim now is to make inference with a Bayesian-like procedure, so we have to deal with an initial assessment consisting of a "prior" φ on a partition $\mathcal{L}$ and a "likelihood function" f related to the set of conditional events $E|H_i$'s, with E an arbitrary event and $H_i \in \mathcal{L}$. This topic has been deeply discussed in [40, 41] by considering several interesting examples.

First of all we need to test the consistency of the global assessment

$$\{f, \varphi\} = \{f(E|H_i), \varphi(A) : H_i \in \mathcal{L}, A \in \langle\mathcal{L}\rangle\}$$

with respect to the framework of reference ($\langle\mathcal{L}\rangle$ denotes the algebra generated by $\mathcal{L}$). The choice of the framework of reference is essentially decided by the prior, since as shown in Theorem 4, a likelihood can be re-read in any framework. This can happen also when the prior comes from a previous inferential process such as the enlargement of an uncertainty assessment (see [15, 22, 28, 41]).

Theorem 6. *Let $\mathcal{L}$ be a partition of Ω, consider a likelihood f related to an event E on $\mathcal{L}$ and consider a probability P, a plausibility Pl and a possibility Π on the algebra $\langle\mathcal{L}\rangle$. Then, the following conditions hold:*

a) the global assessment $\{f, P\}$ is a coherent conditional probability;

b) the global assessment $\{f, Pl\}$ is a coherent conditional plausibility;

c) the global assessment $\{f, \Pi\}$ is a coherent T-conditional possibility (for every continuous t-norm T);

Proof. Condition *a)* has been proved in [39], while condition *c)* has been proved in [1].

Concerning condition *b)* note that Pl on $\langle\mathcal{L}\rangle$ defines a unique basic assignment function m_0 on $\langle\mathcal{L}\rangle$ that is the unique solution of S^0_{Pl} concerning the coherence of Pl. Then, we need to establish whether the assessment $\{f, Pl\}$ is coherent inside conditional plausibility, so we need to check whether the relevant system $S^0_{Pl,f}$ has solution and so whether there is a class of basic assignment $\{m'_\alpha\}$ on $\langle E, \mathcal{L}\rangle$. Notice if the system $S^0_{Pl,f}$ has a solution then coherence with respect to conditional plausibility follows from Theorem 5.

Actually, the atoms in $\langle E, \mathcal{L}\rangle$ are all the events $E \wedge H_i, E^c \wedge H_i$ with $H_i \in \mathcal{L}$. From [18] any plausibility on $\langle\mathcal{L}\rangle$ induces a unique function, called basic plausibility assignment, ν (possibly taking also negative values) on $\langle\mathcal{L}\rangle$ such that $\sum_{A\in\langle\mathcal{L}\rangle} \nu(A) = 1$ and $\sum_{A\in\langle\mathcal{L}\rangle:A\subseteq B} \nu(A) = Pl(B)$.

Let μ be on $\langle\mathcal{L}\rangle$ be the plausibility assignment induced by Pl, consider μ' defined on $\langle E, L\rangle$ as $\mu'(H_i) = 0$, $\mu'(E \wedge H_i) = f(E|H_i)Pl(H_i)$, $\mu'(E^c \wedge H_i) = \mu(H_i) - \mu'(E \wedge H_i)$, and, for any $A \in \langle\mathcal{L}\rangle \setminus \mathcal{L}$, $\mu(A) = \mu'(A)$. By construction $\sum_{A\in\langle E,\mathcal{L}\rangle} \mu'(A) = 1$. For any B in $\langle E, \mathcal{L}\rangle$, but not in $(\langle\mathcal{L}\rangle \cup \{E \wedge H_i, E^c \wedge H_i : H_i \in \mathcal{L}\})$ one has $\mu'(B) = 0$. Then, the function f on $\langle E, L\rangle$ defined as $\sum_{A\in\langle E,\mathcal{L}\rangle:A\subseteq B} \mu'(A) = f(B)$ is such that by construction, for any $B \in \langle\mathcal{L}\rangle$,

$$f(B) = \sum_{A\in\langle E,\mathcal{L}\rangle:A\subseteq B} \mu'(A) =$$

$$\sum_{A\in\langle\mathcal{L}\rangle:A\subseteq B} \mu'(E \wedge A) + \mu'(E^c \wedge A) + \mu'(A) =$$

$$\sum_{A\in\langle\mathcal{L}\rangle:A\subseteq B} \mu(A) = Pl(B)$$

then f extends Pl.

We need to prove that f is a plausibility: the proof can be made by induction, we prove here that is 2-alternating, the proof that it is n-alternating under the hyphothesis that is $(n-1)$-alternating is similar.

For any event $A \in \langle E, \mathcal{L}\rangle$ there is an event $\bar{A} \in \langle\mathcal{L}\rangle$ such that $\bar{A} \subseteq A$ and no event $B \in \langle\mathcal{L}\rangle$ such that $\bar{A} \subset B \subseteq A$, that is the maximal event of $\langle\mathcal{L}\rangle$ contained in A. Then, given any pair of events $A, B \in \langle E, \mathcal{L}\rangle$ let $\bar{A}, \bar{B} \in \langle\mathcal{L}\rangle$ be the two maximal events contained, respectively in A and B. Thus,

$$f(A\vee B) = \sum_{C\in\langle E,\mathcal{L}\rangle:C\subseteq A\vee B} \mu'(C) = \sum_{E\wedge H_i\subseteq A\vee B} \mu'(E\wedge H_i)+$$

$$\sum_{E^c\wedge H_i\subseteq A\vee B} \mu'(E^c \wedge H_i) + \sum_{C\in\langle\mathcal{L}\rangle\setminus\mathcal{L}, C\subseteq A\vee B} \mu'(C) =$$

$$\sum_{H_i\subseteq A\vee B} \mu(H_i) + \sum_{E\wedge H_i\subseteq A\vee B, E^c\wedge H_i\not\subseteq A\vee B} \mu'(E \wedge H_i)+$$

$$\sum_{E^c\wedge H_i\subseteq A\vee B, E\wedge H_i\not\subseteq A\vee B} \mu'(E^c \wedge H_i) + \sum_{C\in\langle\mathcal{L}\rangle\setminus\mathcal{L}, C\subseteq A\vee B} \mu(C)$$

$$= Pl(\overline{A \vee B}) + \sum_{E\wedge H_i\subseteq A\vee B, E^c\wedge H_i\not\subseteq A\vee B} \mu'(E \wedge H_i)+$$

$$\sum_{E^c\wedge H_i\subseteq A\vee B, E\wedge H_i\not\subseteq A\vee B} \mu'(E^c \wedge H_i) =$$

$$Pl(\bar{A} \vee \bar{B}) + \sum_{H_i\subseteq\overline{A\vee B}, H_i\not\subseteq\bar{A}\vee\bar{B}} \mu(H_i)+$$

$$\sum_{E\wedge H_i\subseteq A\vee B, E^c\wedge H_i\not\subseteq A\vee B} \mu'(E \wedge H_i)+$$

$$\sum_{E^c\wedge H_i\subseteq A\vee B, E\wedge H_i\not\subseteq A\vee B} \mu'(E^c \wedge H_i).$$

Note that $A = \bar{A} \vee \bigvee_{H_i\in\mathcal{L}:H_i\not\subseteq A}((E\wedge H_i\wedge A)\vee(E^c\wedge H_i\wedge A))$ and analogously for B. Obviously, $\bar{A} \vee \bar{B} \subseteq A \vee B$ and $\bar{A} \wedge \bar{B}$ coincides with $\overline{A \wedge B}$. Moreover, $\bar{A} \vee \bar{B}$ is included into $\overline{A \vee B}$, but does not coincide with it, in fact $H_i \in \mathcal{L}$ could be included in $A \vee B$, but H_i is not included neither in A nor in B (e.g. $E \wedge H_i \subseteq A$ and $E^c \wedge H_i \subseteq B$). Hence,

$$f(A\vee B) \le Pl(\bar{A})+Pl(\bar{B})-Pl(\bar{A}\wedge\bar{B})+ \sum_{H_i\subseteq A\bar{\vee}B, H_i\not\subseteq\bar{A}\vee\bar{B}} \mu(H_i)+$$

$$\sum_{E\wedge H_i\subseteq A\vee B, E^c\wedge H_i\not\subseteq A\vee B} \mu'(E \wedge H_i)+$$

$$\sum_{E^c\wedge H_i\subseteq A\vee B, E\wedge H_i\not\subseteq A\vee B} \mu'(E^c \wedge H_i)$$

$$\le Pl(\bar{A}) + Pl(\bar{B}) - Pl(\bar{A} \wedge \bar{B})+$$

$$\sum_{H_i\subseteq\overline{A\vee B}, H_i\not\subseteq\bar{A}\vee\bar{B}} (\mu'(E \wedge H_i) + \mu'(E^c \wedge H_i))+$$

$$\sum_{E\wedge H_i\subseteq A\vee B, E^c\wedge H_i\not\subseteq A\vee B} \mu'(E \wedge H_i)+$$

$$\sum_{E^c\wedge H_i\subseteq A\vee B, E\wedge H_i\not\subseteq A\vee B} \mu'(E^c\wedge H_i)$$
$$= f(A)+f(B)-Pl(\bar{A}\wedge\bar{B})$$
$$-\sum_{E\wedge H_i\subseteq A\wedge B, E^c\wedge H_i\not\subseteq A\vee B} \mu'(E\wedge H_i)$$
$$-\sum_{E^c\wedge H_i\subseteq A\wedge B, E\wedge H_i\not\subseteq A\vee B} \mu'(E^c\wedge H_i)$$
$$= f(A)+f(B)-f(A\wedge B)$$

Finally, f induces a conditional plausibility, that we continue to denote by f, on $\langle E,\mathcal{L}\rangle\times\mathcal{H}$ where H is the additive set generated by $H_i\in\mathcal{L}$ such that $f(H_i)>0$. For any $H_i\in\mathcal{L}$ one has
$f(E|H_i)=\frac{f(E\wedge H_i)}{f(H_i)}=\frac{\mu'(E\wedge H_i)}{Pl(H_i)}=f(E|H_i)$.
This implies that the system $S^0_{Pl,f}$ admits a solution and so for the above consideration the assessment $\{Pl,f\}$ is a coherent conditional plausibility. □

3.2 Aggregated likelihoods

Now we study the properties of *aggregated likelihood functions*, that is all the coherent extensions g of the assessment $\{f(E|H_i)\ :\ H_i\in\mathcal{L}\}$ to the events $E|K$, with K belonging to the additive set $\mathcal{H}=\langle\mathcal{L}\rangle^0=(\langle\mathcal{L}\rangle\setminus\{\emptyset\})$.

The interest derives from inferential problems in which the available information consists of a (probabilistic or plausibilistic or possibilistic) "prior" on a partition $\{K_j\}$ and a likelihood related to the events of another partition refining the previous one. So first of all we need to aggregate the likelihood function preserving coherence with the framework of reference.

In what follows $g:\{E\}\times\mathcal{H}\to[0,1]$ denotes a function such that its restriction to $\{E\}\times\mathcal{L}$ coincides with f.

We recall a common feature of probabilistic and possibility framework: any aggregated likelihood g, regarded as a coherent conditional probability or a coherent T-conditional possibility, satisfies the following condition for every $K\in\mathcal{H}$:

$$\min_{H_i\subseteq K} f(E|H_i)\le g(E|K)\le \max_{H_i\subseteq K} f(E|H_i). \qquad (6)$$

Now the question is to investigate whether an aggregated likelihood seen as a coherent conditional plausibility must satisfy the same constraints.

In the following example we show that, for a coherent conditional plausibility, the value $\max_{H_i\subseteq K} f(E|H_i)$ is not an upper bound.

Example 1. *Let $\mathcal{L}=\{H_1,H_2\}$ be a partition and E an event logically independent of the events $H_i\in\mathcal{L}$. Consider the following likelihood on $\mathcal{L}$*

$$f(E|H_1)=\frac{1}{4};\ f(E|H_2)=\frac{1}{2}$$

and let g be a function extending f on $\{E\}\times\mathcal{H}$ such that $g(E|H_1\vee H_2)=\frac{3}{4}=f(E|H_1)+f(E|H_2)$.

From equation (6) it follows that g is not a coherent T-conditional possibility or conditional probability; we prove that it is indeed a coherent conditional plausibility. For that let us consider the following system with unknowns $m_0(C)$, where $C\in\langle E,\mathcal{L}\rangle$

$$(S^0)=\begin{cases} 1/4\cdot\sum_{H_1\wedge C\neq\emptyset} m_0(C)=\sum_{H_1\wedge E\wedge C\neq\emptyset} m_0(C),\\ 1/2\cdot\sum_{H_2\wedge C\neq\emptyset} m_0(C)=\sum_{H_2\wedge E\wedge C\neq\emptyset} m_0(C),\\ 3/4\cdot\sum_{(H_1\vee H_2)\wedge C\neq\emptyset} m_0(C)=\sum_{(H_1\vee H_2)\wedge E\wedge C\neq\emptyset} m_0(C),\\ \sum_{C\subseteq H_1\vee H_2} m_0(C)=1\\ m_0(C)\ge 0, \qquad \forall C\in\langle E,\mathcal{L}\rangle\end{cases}$$

It is easy to see that the basic assignment:

$$m_0((E\wedge H_1)\vee(E^c\wedge H_2))=m_0(H_1\vee(E^c\wedge H_2))=\frac{1}{8},$$
$$m_0((E^c\wedge H_1)\vee(E\wedge H_2))=m_0((E^c\wedge H_1)\vee H_2)=$$
$$m_0(E^c\wedge(H_1\vee H_2))=\frac{1}{4}$$

and $m_0(C)=0$ for any other event $C\in\langle E,\mathcal{L}\rangle$, is a solution of S_0, giving positive plausibility to both the events H_i.

The following example shows that also the lower bound of condition (6) can be violated in the plausibility framework.

Example 2. *Let $\mathcal{L}=\{H_1,H_2\}$ be a partition and E an event logically independent of all the events H_i.*

Consider the following aggregated likelihood on $\mathcal{H}$

$$f(E|H_1)=f(E|H_2)=\frac{2}{3},\ f(E|H_1\vee H_2)=\frac{1}{2}.$$

To prove that the assessment is coherent within a conditional plausibility, we consider the following system with unknowns $m_0(C)$, where $C\in\langle E,\mathcal{L}\rangle$

$$(S^0)=\begin{cases} 2/3\cdot\sum_{H_1\wedge C\neq\emptyset} m_0(C)=\sum_{H_1\wedge E\wedge C\neq\emptyset} m_0(C),\\ 2/3\cdot\sum_{H_2\wedge C\neq\emptyset} m_0(C)=\sum_{H_2\wedge E\wedge C\neq\emptyset} m_0(C),\\ 1/2\cdot\sum_{(H_1\vee H_2)\wedge C\neq\emptyset} m_0(C)=\sum_{(H_1\vee H_2)\wedge E\wedge C\neq\emptyset} m_0(C),\\ \sum_{C\subseteq H_1\vee H_2} m_0(C)=1\\ m_0(C)\ge 0, \qquad \forall C\in\langle E,\mathcal{L}\rangle\end{cases}$$

The following basic assignment on $\langle E,\mathcal{L}\rangle$:

$$m_0=(E^c\wedge H_1)=m_0(E^c\wedge H_2)=m_0(E)=m_0(\Omega)=\frac{1}{4}$$

and $m_0(C) = 0$ for any other event $C \in \langle E, \mathcal{L}\rangle$, is a solution of S_0, giving positive plausibility to both the events H_i.

The fact that the lower bound of coherent values of $Pl(E|H_i \vee H_j)$ can be less than $\inf\{Pl(E|H_i), Pl(E|H_j)\}$ is an indirect proof that a conditional plausibility (Definition 2) is not an upper envelope of a set of conditional probabilities.

Theorem 7. *Any coherent conditional plausibility Pl, extending a likelihood $f : E \times \mathcal{L} \to [0,1]$ on $E \times \mathcal{H}$, satisfies the following inequality for every $K \in \mathcal{H}$:*

$$(L2) \quad 0 \leq Pl(E|K) \leq \min\{\sum_{H_i \subseteq K} f(E|H_i), 1\}.$$

Proof. Since f is a coherent conditional plausibility assessment, then there is a coherent conditional plausibility Pl on $\mathcal{B} \times \mathcal{H}$ with $\mathcal{B} = \langle \mathcal{H} \cup \{E\}\rangle$, extending f. The restriction of Pl to $E \times \mathcal{H}$ is a coherent conditional plausibility and for every $K \in \mathcal{H}$, satisfies (3) and $Pl(E|K) \geq 0$. So we have $0 \leq g(E|K) \leq \sum_{H_i \subseteq K} f(E|H_i) g(H_i|K)$, and then the thesis. □

Theorem 7 shows that in plausibility framework there is much more freedom than in both probabilistic and possibilistic ones, where aggregated likelihood functions are monotone, with respect to $\subseteq$, only if the extension is obtained, for every K, as $\max_{H_i \subseteq K} f(E|H_i)$ and they are anti-monotone if and only if their extensions are obtained as $\min_{H_i \subseteq K} f(E|H_i)$.

Since any likelihood (see Theorem 4) is also a coherent conditional probability and in [10, 12] it is proved that an aggregated likelihood coherent within conditional probability can be obtained by taking the minimum (maximum), this extension is obviously also a coherent conditional plausibility.

In the following Proposition we prove that we could take the sum of likelihoods.

Theorem 8. *Let f be a likelihood on $\mathcal{L}$ related to an event E and consider the function g on $\{E\} \times \mathcal{H}$ defined as follows: for all $K_1, K_2 \in \mathcal{H}$ with $K_1 \wedge K_2 = \emptyset$*

$$g(E|K_1 \vee K_2) = g(E|K_1) + g(E|K_2).$$

If $\sum_{H_i \in \mathcal{L}} f(E|H_i) \leq 1$, then g

is a coherent conditional plausibility extending f.

Proof. To prove the result it is enough to consider the following basic assignment m on $\langle E, \mathcal{L}\rangle$:

$$m((E \wedge H_i) \vee \bigvee_{j \neq i} (E^c \wedge H_j)) + m(H_i \vee \bigvee_{j \neq i} (E^c \wedge H_j)) = f(E|H_i)$$

for $H_i \in \mathcal{L}$ and $m(E^c) = 1 - \sum_{H_i \in \mathcal{L}} f(E|H_i)$.

It is easy to show that this basic assignment m is agreeing with g (see Theorem 3) and the plausibility of H_i is positive. □

4 Fuzzy sets

The aim of this sections is to apply the results of the previous section to an inferential problem, starting from linguistic information (fuzzy sets) and statistical information. We refer to the interpretation of fuzzy sets in terms of coherent conditional probabilities [8, 9, 5]: the idea behind such interpretation is related to that given in the seminal work [32], and we extend it inside imprecise probabilities.

Let X be a (not necessarily numerical) variable, with range $\mathcal{C}_X$, and, for any $x \in \mathcal{C}_X$, let us indicate by A_x the event $\{X = x\}$. Let φ be any *property* related to the variable X and let us refer to the state of information of a real (or fictitious) person that will be denoted by "You". A coherent conditional probability (possibility) [plausibility] $f(E_\varphi|A_x)$ measures (in different frameworks) the degree of belief of You in E_φ, when X assumes the different values x in $\mathcal{C}_X$.

Then $f(E_\varphi|\cdot)$ comes out to be a natural interpretation of the membership function $\mu_\varphi(\cdot)$, analogously to the probabilistic case [9] (see also [8, 5]).

Definition 4. *For any variable X with range $\mathcal{C}_X$ and a related property φ, the fuzzy subset E^*_φ of $\mathcal{C}_X$ is the pair*

$$E^*_\varphi = \{E_\varphi\, ,\, \mu_{E_\varphi}\},$$

with $\mu_{E_\varphi}(x) = f(E_\varphi|A_x)$ for every $x \in \mathcal{C}_X$ (f stands for a coherent conditional probability or plausibility or possibility).

Theorem 4 assures that any assessment $\{f(E|A_x)\}_{x \in \mathcal{C}_X}$ is coherent within conditional probability, plausibility and possibility: so we have no syntactical restriction for f; Theorem 5 assures that in all the three frameworks the notion of fuzzy subsets, defined by a likelihood, is a generalization of crisp subsets.

Now denote by $\varphi \vee \psi$, $\varphi \wedge \psi$, respectively, the properties "φ or ψ" , "φ and ψ", and define

$$E_{\varphi \vee \psi} = E_\varphi \vee E_\psi\ ,$$

$$E_{\varphi \wedge \psi} = E_\varphi \wedge E_\psi\ .$$

Let us consider two fuzzy subsets E^*_φ, E^*_ψ, corresponding to the same variable X, with the events

E_φ, E_ψ logically independent with respect to X. As proved in [9], for any given x in the range of X, the assessment $P(E_\varphi \wedge E_\psi | A_x) = v$ is coherent within a conditional probability if and only if takes values in the interval

$$\max\{P(E_\varphi|A_x) + P(E_\psi|A_x) - 1, 0\} \leq v \leq \leq \min\{P(E_\varphi|A_x), P(E_\psi|A_x)\}.$$

It is easy to see that the assessment $f(E_\varphi \wedge E_\psi|A_x) = v$ is coherent within a conditional plausibility or possibility if and only if takes values in the interval

$$0 \leq v \leq \min\{f(E_\varphi|A_x), f(E_\psi|A_x)\}.$$

Then, the lower bound of conditional probability does not continue to be valid.

While probability rules imply that given a value to $f(E_\varphi \wedge E_\psi|A_x)$, we get also the value of $f(E_\varphi \vee E_\psi|A_x)$, in the case of possibility we have that the value of $f(E_\varphi \vee E_\psi|A_x)$ is univocally determined by $f(E_\varphi|A_x)$ and $f(E_\psi|A_x)$ without taking into account the value of $f(E_\varphi \wedge E_\psi|A_x)$.

In the case of plausibility we have that the value of $f(E_\varphi \vee E_\psi|A_x)$ is not univocally determined but it must be

$$\max\{f(E_\varphi|A_x), f(E_\psi|A_x)\} \leq f(E_\varphi \vee E_\psi|A_x) \leq$$

$$\min\{f(E_\varphi|A_x) + f(E_\psi|A_x) - f(E_\varphi \wedge E_\psi|A_x), 1\}$$

Then we can put

$$E^*_\varphi \cup E^*_\psi = \{E_{\varphi\vee\psi}\, ,\, \mu_{\varphi\vee\psi}\}\ ,$$

$$E^*_\varphi \cap E^*_\psi = \{E_{\varphi\wedge\psi}\, ,\, \mu_{\varphi\wedge\psi}\}\ ,$$

with

$$\mu_{\varphi\vee\psi}(x) = f(E_\varphi \vee E_\psi|A_x)\ ,$$

$$\mu_{\varphi\wedge\psi}(x) = f(E_\varphi \wedge E_\psi|A_x)\,.$$

Moreover, denoting by $E^*_{\neg\varphi}$ the complementary fuzzy set of E^*_φ, the relation $E_{\neg\varphi} \neq (E_\varphi)^c$ holds, since the propositions "You *claim* $\neg\varphi$" and "You *do not claim* φ" are logically independent. In fact, we can claim both "X has the property φ" and "X has the property $\neg\varphi''$, or only one of them or finally neither of them; similarly are logical independent E_φ and E_ψ, where ψ is the superlative of φ.

Then, while $E_\varphi \vee (E_\varphi)^c = C_X$, we have instead $E_\varphi \vee E_{\neg\varphi} \subset C_X$, and, if we consider the union of a fuzzy subset and its complement

$$E^*_\varphi \cup (E^*_\varphi)' = \{E_{\varphi\vee\neg\varphi}\, ,\, \mu_{\varphi\vee\neg\varphi}\}$$

we obtain in general a *fuzzy subset* of C_X.

The constraints on the function f depend, as shown before, on the framework of reference.

The concept of fuzzy event, as introduced by Zadeh, can be seen an ordinary event of the kind

$$E_\varphi = \text{"You claim that } X \text{ is } \varphi\text{"}.$$

and for any uncertainty measure (probability, possibility and plausibility) on the events related to X the assessment together μ_φ is coherent with respect the relative measure (see Theorem 6) and so coherently extendible to E_φ (Theorem 2 for plausibilities, [17] for conditional possibilities).

In the case of probability and possibility it is easily to see that the only coherent value for the probability or possibility of E_φ is

$$g(E_\varphi) = \bigoplus_{x\in\mathcal{C}_X} \mu_{\varphi_i}(x) \bigodot g(x)\,,$$

where $\bigoplus$ and $\bigodot$ are the sum and the product in the case of probability, while they are the maximum and minimum in the case of possibility.

Obviously, only in the case of probability it coincides with Zadeh's definition of the probability of a "fuzzy event" [42].

5 Conclusion

The first part of the paper is devoted into studying likelihood functions seen as assessment on a set of conditional events $E|H_i$, with E the evidence and H_i varying on a partition $\mathcal{L}$. It is shown that likelihood functions are assessment coherent with respect probability, possibility and plausibility. Then, inferential processes, like Bayesian one, are studied in the different setting taking a likelihood function and a prior, that could be a probability or a possibility or a plausibility. I particular we prove that any likelihood function on $E \times \mathcal{L}$ and any plausibility on $\mathcal{L}$, with $\mathcal{L}$ a partition, are globally coherent within conditional plausibility. Then, a comparison of aggregated likelihoods, that are coherent extensions of a likelihood function on $E \times \mathcal{L}$ to $E \times \langle\mathcal{L}\rangle^0$ is studied in the different setting by showing the common characteristic and the specific features.

Finally, by using the above results we give an interpretation of fuzzy sets in terms of likelihood function in the different setting: by starting from the interpretation in the probabilistic setting given in [9] we give a similar interpretation in plausibility and possibilistic settings.

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