Bayesian-like inference, complete disintegrability and complete conglomerability in coherent conditional possibility theory

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Abstract

In this paper we consider Bayesian-like inference processes involving coherent T-conditional possibilities assessed on infinite sets of conditional events. For this, a characterization of coherent assessments of possibilistic prior and likelihood is carried on. Since we are working in a finitely maxitive setting, the notions of complete disintegrability and of complete conglomerability are also studied and their relevance in the infinite version of the possibilistic Bayes formula is highlighted.

Keywords. Complete disintegrability, complete conglomerability, finite maxitivity, T-conditional possibility, possibilistic likelihood function, coherence.

1 Introduction

This paper deals with finitely maxitive T-conditional possibilities (with T any continuous t-norm) and focuses the attention on problems related to the updating of possibility by Bayesian-like procedures.

In the first part of the paper we mainly deal with the characterization of coherent T-conditional possibility assessments, both for arbitrary families of conditional events and for particular families of the type $\{H_i, E|H_i\}_{i\in I}$, with I infinite, where the H_i's form a partition of the sure event while E is an arbitrary event. For these last assessments we also characterize the set of coherent values for their extension to E, in the case T is the minimum or a strict t-norm and E is logically independent of the H_i's.

In the second part we take into consideration two concepts: complete disintegrability and complete conglomerability for events, defined in analogy to those introduced in probability theory (originally given for countable partitions [18, 29, 1]), considering infinite partitions with arbitrary cardinality. As it is well-known, in probability theory the two properties (see, e.g., [17, 21, 29, 30, 31, 4]) are strictly related to σ-additivity. In fact for finitely additive conditional probabilities it is possible to have examples which, contrary to intuition, show that a P needs not be conglomerative (and so disintegrable). In Bayesian literature, the phenomenon of nonconglomerability has emerged in the so-called marginalization paradoxes [7]. In this paper we show similarities and differences between the probabilistic and possibilistic contexts about complete disintegrability and complete conglomerability, moreover we investigate their connection with complete maxitivity. In particular, we find that, for a fixed infinite partition $\mathcal{L}$, complete disintegrability w.r.t. $\mathcal{L}$ implies both complete maxitivity w.r.t. $\mathcal{L}$ and complete conglomerability w.r.t. $\mathcal{L}$ but the implications are not invertible. Furthermore, complete conglomerability w.r.t. $\mathcal{L}$ and complete maxitivity w.r.t. $\mathcal{L}$ are independent.

2 Coherent T-conditional possibility

In this section we recall the definition of conditional possibility given in [5, 6, 13, 14], that can be obtained as a particular instance of the one introduced in [10].

An *event* E is singled out by a Boolean proposition, that is a statement that can be either true or false. Since in general it is not known whether E is true or not, we are uncertain on E, which is said to be *possible*. Two particular events are the *certain event* Ω and the *impossible event* $\emptyset$, that coincide with, respectively, the top and the bottom of every Boolean algebra $\mathcal{B}$ of events, i.e., a set of events closed w.r.t. the familiar Boolean operations of *contrary* ^{c}, *conjunction* $\wedge$ and *disjunction* $\vee$ and equipped with the partial order $\subseteq$. Recall that due to Stone's theorem, events can be represented as subsets of a universe set that is identified with Ω: in this case we continue to use ^{c}, $\wedge$ and $\vee$ in place of set-theoretic operations.

A *conditional event* $E|H$ is an ordered pair (E, H), with $H \neq \emptyset$, where E and H are events of the same "nature", but with a different role (in fact H acts as a "possible hypothesis"). In particular any event E can be seen as the conditional event $E|\Omega$.

In what follows, $\mathcal{B} \times \mathcal{H}$ denotes a set of conditional events with $\mathcal{B}$ a Boolean algebra and $\mathcal{H}$ an additive set (i.e., closed with respect to finite disjunctions) such that $\mathcal{H} \subseteq \mathcal{B}^0 = \mathcal{B} \setminus \{\emptyset\}$. Moreover, given an arbitrary set $\mathcal{G} = \{E_j|H_j\}_{j\in J}$, denote with $\langle\{E_j, H_j\}_{j\in J}\rangle$ the Boolean algebra generated by the events $\{E_j, H_j\}_{j\in J}$.

We recall that a *t-norm* T is a commutative, associative, increasing, binary operation on $[0,1]$, having 1 as neutral element. A t-norm is called *continuous* (analogously, *left-continuous* or *right-continuous*) if it is continuous as a function, in the usual interval topology on $[0,1]^2$. Prototypical examples of continuous t-norms are the minimum, the algebraic product and the Łukasiewicz t-norm, moreover, any continuous t-norm is isomorphic to an ordinal sum of previous t-norms (see for instance [24]). A t-norm is called *strict* if it is continuous and strictly monotone: strict t-norms are isomorphic to the algebraic product through an order automorphism of the unit interval.

Definition 1. *Let T be any t-norm. A function $\Pi : \mathcal{B} \times \mathcal{H} \to [0,1]$ is a* **T-conditional possibility** *if it satisfies the following properties:*

(i) $\Pi(E|H) = \Pi(E \wedge H|H)$*, for every* $E \in \mathcal{B}$ *and* $H \in \mathcal{H}$*;*

(ii) $\Pi(\cdot|H)$ *is a finitely maxitive possibility on* $\mathcal{B}$*, for any* $H \in \mathcal{H}$*;*

(iii) $\Pi(E \wedge F|H) = T(\Pi(E|H), \Pi(F|E \wedge H))$*, for any* $H, E \wedge H \in \mathcal{H}$ *and* $E, F \in \mathcal{B}$*.*

Let us stress that condition *(ii)* requires that, for every $H \in \mathcal{H}$, $\Pi(\emptyset|H) = 0$, $\Pi(\Omega|H) = 1$ and for every $E_1, \ldots, E_n \in \mathcal{B}$, $\Pi\left(\bigvee_{i=1}^n E_i|H\right) = \max\limits_{i=1,\ldots,n} \Pi(E_i|H)$, which is called *finite maxitivity axiom* [33]. Moreover conditions *(i)* and *(ii)* imply that $\Pi(H|H) = 1$ for every $H \in \mathcal{H}$.

Notice that in this paper we do not postulate the stronger condition of *complete maxitivity*, which requires that for every $\{E_i\}_{i\in I} \subseteq \mathcal{B}$ with $\bigvee_{i\in I} E_i \in \mathcal{B}$ and arbitrary I, $\Pi\left(\bigvee_{i\in I} E_i|H\right) = \sup\limits_{i\in I} \Pi(E_i|H)$, thus we always mean finitely maxitive T-conditional possibilities even when not explicitly stated.

Remark 1. *Every finitely maxitive unconditional possibility* $\Pi(\cdot)$ *on* $\mathcal{B}$ *can be seen as a T-conditional possibility on* $\mathcal{B} \times \{\Omega\}$*, where T is an arbitrary t-norm. In particular, for a T-conditional possibility* Π *on* $\mathcal{B} \times \mathcal{H}$*, we will write* $\Pi(E)$ *for* $\Pi(E|\Omega)$*, provided that* $\Omega \in \mathcal{H}$*.*

For every finite set of incompatible events $H_1, \ldots, H_n \in \mathcal{H}$ with $H = \bigvee_{i=1}^n H_i$ and for every $E \in \mathcal{B}$, axioms *(ii)* and *(iii)* imply a possibilistic counterpart of the well-known *disintegration formula*

$$\Pi(E|H) = \max_{i=1,\ldots,n} \{T(\Pi(E|H_i), \Pi(H_i|H))\}. \qquad (1)$$

Definition 1 does not require any particular property for the t-norm T. The only constraint is the distributivity over the maximum operation used in condition *(ii)*, but this constraint is vacuous since every t-norm is distributive over max.

Nevertheless, continuity of the t-norm T is fundamental [14, 27] in order to guarantee the extendability (generally not in a unique way) of a T-conditional possibility on $\mathcal{B} \times \mathcal{H}$ to a *full* T-conditional possibility on $\mathcal{B}$ (i.e., with domain $\mathcal{B} \times \mathcal{B}^0$). For this, in the rest of the paper we will always assume T is continuous when not explicitly stated.

Differently from other common notions of conditioning in possibility theory [36, 23, 22, 15], a full T-conditional possibility $\Pi(\cdot|\cdot)$ is not singled out by a single unconditional possibility measure $\Pi(\cdot)$, in general, but one needs a class of finitely maxitive measures [33] defined on a family of ideals linearly ordered by proper set inclusion.

Remark 2. *We notice that in the particular case where the t-norm T is the usual product,* $\Omega \in \mathcal{H}$ *and* $\Pi(H) = \Pi(H|\Omega) > 0$*, for every* $H \in \mathcal{H}$*, the definition of T-conditional possibility coincides with Dempster's rule [20]:*

$$\Pi_D(E|H) = \frac{\Pi(E \wedge H)}{\Pi(H)}.$$

We recall that the conditional possibility Π_D *is not necessarily a coherent conditional upper probability (see [16, 35]), vice versa a conditional possibility obtained as upper envelope of a class of conditional probabilities in general does not satisfy condition (iii) of Definition 1.*

Definition 2. *Let* $\mathcal{B}$ *be a Boolean algebra and T a continuous t-norm. A family* $\{(\mathcal{I}_i, \pi_i) : i \in I\}$ *is a* **T-nested class** *if:*

(a) for every $i \in I$*,* $\mathcal{I}_i$ *is a Boolean ideal of* $\mathcal{B}$ *and the family* $\{\mathcal{I}_i : i \in I\}$ *is linearly ordered by proper set inclusion;*

(b) for every $E \in \mathcal{B}^0$*, there exists* $i \in I$ *such that* $E \in \mathcal{I}_i \setminus \bigcup\{\mathcal{I}_j : \mathcal{I}_j \subset \mathcal{I}_i\}$*;*

(c) for every $i \in I$*,* π_i *is a (non-identically equal to 0) finitely maxitive measure on* $\mathcal{I}_i$ *ranging in* $[0,1]$*, such that for every* $E \in \mathcal{I}_i$*,* $\pi_i(E) < 1$ *if and only if* $E \in \bigcup\{\mathcal{I}_j : \mathcal{I}_j \subset \mathcal{I}_i\}$*;*

(d) *for every $i,j \in I$ such that $\mathcal{I}_i \subset \mathcal{I}_j$ and every $E,F \in \mathcal{I}_i$, all the solutions of equation $\pi_i(E \wedge F) = T(x,\pi_i(F))$ are solutions of the equation $\pi_j(E \wedge F) = T(x,\pi_j(F))$;*

(e) *for every $i,j \in I$ such that $\mathcal{I}_i \subset \mathcal{I}_j$, $\pi_{j|\mathcal{I}_i} \leq \pi_i$.*

Notice that, Definition 2 is equivalent in the finite case to the notion of T-nested class introduced in [14]. In particular, each finitely maxitive measure π_i on $\mathcal{I}_i$ is a restriction of a finitely maxitive possibility measure on $\mathcal{B}$.

The algebraic requirement on the domain of the function Π in Definition 1 cannot be relaxed, indeed axioms *(i)*–*(iii)* are no more sufficient to characterize Π if it is defined on an arbitrary set of conditional events $\mathcal{G}$. Hence, in order to deal with this eventuality, the axiomatic system must be reinforced going back to the concept of *coherence*, originally introduced by de Finetti [19] in the context of (finitely additive) probabilities.

Definition 3. *Let T be any continuous t-norm. A function $\Pi : \mathcal{G} \to [0,1]$ is a* **coherent T-conditional possibility (assessment)** *if there exists a T-conditional possibility $\Pi' : \mathcal{B}\times\mathcal{H} \to [0,1]$ such that $\Pi'_{|\mathcal{G}} = \Pi$, where $\mathcal{B}\times\mathcal{H} \supseteq \mathcal{G}$ with $\mathcal{B}$ a Boolean algebra and $\mathcal{H} \subseteq \mathcal{B}^0$ and additive class.*

Remark 3. *Previous definition can be equivalently formulated by requiring that Π can be extended as a full T-conditional possibility on $\mathcal{B}$. In fact in [27] the extendability of any T-conditional possibility on $\mathcal{B}\times\mathcal{H}$ to a full T-conditional possibility on $\mathcal{B}$ has been proved.*

Coherent T-conditional possibility assessments on finite domains have been characterized in [14]. Such characterization has been extended to the infinite case in [27], where the coherence of an assessment Π on $\mathcal{G}$ is expressed in terms of coherence of $\Pi_{|\mathcal{F}}$ on every finite $\mathcal{F} \subseteq \mathcal{G}$. The following Theorem 1 provides also a characterization in terms of a T-nested class agreeing with the assessment.

Theorem 1. *Let T be a continuous t-norm, $\mathcal{G} = \{E_j|H_j\}_{j\in J}$ an arbitrary set of conditional events and $\mathcal{B}$ the Boolean algebra generated by $\{E_j,H_j\}_{j\in J}$. For any $\mathcal{F} = \{E_1|H_1,\ldots,E_n|H_n\} \subseteq \mathcal{G}$, let $\mathcal{B}_\mathcal{F}$ be the Boolean algebra generated by $\{E_i,H_i\}$ whose set of atoms is $\mathcal{C}_\mathcal{F}$, and $\mathcal{H}_\mathcal{F} \subseteq \mathcal{B}^0_\mathcal{F}$ an additive set such that $\{H_i\} \subseteq \mathcal{H}_\mathcal{F}$. For a function $\Pi : \mathcal{G} \to [0,1]$, the following statements are equivalent:*

(i) Π *is a coherent T-conditional possibility on $\mathcal{G}$;*

(ii) *for any $\mathcal{F} = \{E_1|H_1,\ldots,E_n|H_n\} \subseteq \mathcal{G}$, if $\mathcal{C}_{\mathcal{F}0} = \{C_r \in \mathcal{C}_\mathcal{F} : C_r \subseteq H_0^0\}$ and $H_0^0 = \bigvee_{H\in\mathcal{H}_\mathcal{F}} H$, there exists a sequence of compatible systems $\mathcal{S}^\Pi_{\mathcal{F}\alpha}$, for $\alpha = 0,\ldots,k$, with unknowns $x_r^\alpha \geq 0$ for $C_r \in \mathcal{C}_{\mathcal{F}\alpha}$,*

$$\mathcal{S}^\Pi_{\mathcal{F}\alpha} : \begin{cases} \max\limits_{C_r\subseteq E_i\wedge H_i} x_r^\alpha = T\left(\Pi(E_i|H_i), \max\limits_{C_r\subseteq H_i} x_r^\alpha\right) \\ \left[\text{for } E_i|H_i \in \mathcal{F} \text{ s.t. } \max\limits_{C_r\subseteq H_i} \xi_r^{\alpha-1} < 1\right] \\ x_r^\alpha \geq \xi_r^{\alpha-1}, \text{ if } C_r \in \mathcal{C}_{\mathcal{F}\alpha} \\ \xi_r^{\alpha-1} = T\left(x_r^\alpha, \max\limits_{C_s\in\mathcal{C}_{\mathcal{F}\alpha}} \xi_s^{\alpha-1}\right), \text{ if } C_r \in \mathcal{C}_{\mathcal{F}\alpha} \\ \max\limits_{C_r\in\mathcal{C}_{\mathcal{F}\alpha}} x_r^\alpha = 1 \end{cases} \tag{2}$$

where $\overline{\xi}^\alpha$ (with r-th component ξ_r^α) is the solution of the system $\mathcal{S}^\Pi_{\mathcal{F}\alpha}$ and $\mathcal{C}_{\mathcal{F}\alpha}$ is the set of atoms $\{C_r \in \mathcal{C}_{\mathcal{F}\alpha-1} : C_r \subseteq H_0^\alpha\}$ with $H_0^\alpha = \bigvee\left\{H \in \mathcal{H}_\mathcal{F} : \max\limits_{C_r\subseteq H} \xi_r^\beta < 1, \beta \leq \alpha-1\right\}$, moreover $\xi_r^{-1} = 0$ for any C_r in $\mathcal{C}_{\mathcal{F}0}$;

(iii) *there exists a T-nested class $\{(\mathcal{I}_i,\pi_i) : i \in I\}$ on $\mathcal{B}$ such that for every $E_j|H_j \in \mathcal{G}$ there exists $i \in I$ such that $H_j \in \mathcal{I}_i$ and $\pi_i(H_j) = 1$ and $\pi_i(E_j \wedge H_j) = \Pi(E_j|H_j)$.*

Proof. The equivalence between *(i)* and *(ii)* has been proved in [27]. To prove the equivalence between *(i)* and *(iii)* we follow the line of the construction introduced by Krauss in [25] for full conditional probabilities. Due to space limitations we give here just a sketch of the proof. For this aim, consider that for any full T-conditional possibility Π' on $\mathcal{B}$ it is possible to define a total preorder $\preceq$ on $\mathcal{B}^0$, setting $E \preceq F$ if and only if $\Pi'(F|E\vee F) = 1$, for every $E,F \in \mathcal{B}^0$. For every $E \in \mathcal{B}^0$, the relation $\preceq$ determines the Boolean ideal $\mathcal{I}_E = \{F \in \mathcal{B}^0 : F \preceq E\} \cup \{\emptyset\}$, and the family $\{\mathcal{I}_E : E \in \mathcal{B}^0\}$ results to be linearly ordered by set inclusion. For every $E \in \mathcal{B}^0$, define $\pi_E(F) = \Pi'(F|E\vee F)$ for every $F \in \mathcal{I}_E$, which results to be a finitely maxitive measure on the ideal $\mathcal{I}_E$. The family $\{(\mathcal{I}_E,\pi_E) : E \in \mathcal{B}^0\}$ is such that if $\mathcal{I}_E = \mathcal{I}_F$ then $\pi_E = \pi_F$. Thus, up to equal ideals, we can obtain a unique T-nested class $\{(\mathcal{I}_i,\pi_i) : i \in I\}$ which uniquely represents the full T-conditional possibility Π' on $\mathcal{B}$, since for every $E|H \in \mathcal{B}\times\mathcal{B}^0$, there exists an index $i \in I$ such that $\pi_i(H) = 1$ and $\pi_i(E\wedge H) = \Pi'(E|H)$. Now, since by Remark 3 the coherence of the assessment Π is equivalent to the existence of a full T-conditional possibility Π' on $\mathcal{B}$ extending Π, this is equivalent, in turn, to the existence of a T-nested class on $\mathcal{B}$ agreeing with the assessment Π. □

Remark 4. *In condition (ii) of previous theorem, for* *any finite $\mathcal{F} \subseteq \mathcal{G}$, the sequence of solutions $\overline{\xi}^0, \dots, \overline{\xi}^k$ gives rise to a class of possibilities $\mathcal{P}^\Pi = \{\Pi_0, \dots, \Pi_k\}$ on $\mathcal{B}_\mathcal{F}$ representing a T-conditional possibility on $\mathcal{B}_\mathcal{F} \times \mathcal{H}_\mathcal{F}$ extending $\Pi_{|\mathcal{F}}$ [27]. The choice of $\mathcal{H}_\mathcal{F}$ essentially impacts on the number of systems to solve [2, 3]. Let us notice that for the sake of convenience one can always take for $\mathcal{H}_\mathcal{F}$ the minimal additive set containing $\{H_i\}$, that is, the additive set generated by the H_i's. In the particular case $\mathcal{H}_\mathcal{F}$ is taken equal to $\mathcal{B}^0_\mathcal{F}$, then the solutions $\overline{\xi}^0, \dots, \overline{\xi}^k$ correspond exactly to a finite T-nested class $\{(\mathcal{I}_0, \pi_0), \dots, (\mathcal{I}_k, \pi_k)\}$ with $\mathcal{I}_\alpha \subset \mathcal{I}_{\alpha-1}$, $\alpha = 1, \dots, k$.*

Remark 5. *The characterization of coherence given in Theorem 1 implies that if $\Pi : \mathcal{G}' \to [0,1]$ is coherent, then for any subset $\mathcal{G} \subset \mathcal{G}'$ also $\Pi_{|\mathcal{G}}$ is coherent.*

Now we focus on the main t-norms used for conditioning in possibility theory, i.e., the minimum and strict t-norms. Under this choice, the coherence of an assessment is a sufficient (and necessary) condition for the extendability to any superset of conditional events, as stated in next theorem [27], which is a possibilistic counterpart of the celebrated de Finetti's fundamental theorem for conditional probabilities.

Theorem 2. *Let T be the minimum or a strict t-norm. Let $\mathcal{G}$ be an arbitrary set of conditional events and $\Pi : \mathcal{G} \to [0,1]$ a coherent T-conditional possibility. Then Π can be extended as a coherent T-conditional possibility Π' to any superset $\mathcal{G}' \supset \mathcal{G}$. Moreover, if $\mathcal{G}' = \mathcal{G} \cup \{E|H\}$ then the coherent values for $\Pi'(E|H)$ lie in a closed interval $[\pi_*, \pi^*]$.*

Previous theorem, whose proof relies on Zorn's lemma, generalizes to the infinite case a result proved in [14] for finite domains. In particular, the extension interval $[\pi_*, \pi^*]$ is computed as the intersection of all the intervals $[\pi_{\mathcal{F}*}, \pi_\mathcal{F}{}^*]$ expressing the coherent extensions of $\Pi_{|\mathcal{F}}$ on $E|H$, for any finite subfamily $\mathcal{F} \subseteq \mathcal{G}$.

Remark 6. *Let $\Pi : \mathcal{G}' \to [0,1]$ be a coherent T-conditional possibility and $\mathcal{G} \subset \mathcal{G}'$. If we denote with $[\pi'_*, \pi'^*]$ the extension interval of Π on $E|H$ and with $[\pi_*, \pi^*]$ the extension interval of $\Pi_{|\mathcal{G}}$ on $E|H$, then it holds $[\pi'_*, \pi'^*] \subseteq [\pi_*, \pi^*]$.*

Example 1. *Take $\mathbb{N}$ as universe, let $\mathcal{E} = \{E_i = \{i\}\}_{i \in \mathbb{N}}$, and $\mathcal{H} = \{H_1 = \{1\}^c, \mathbb{N}\}$. Consider the assessment Π defined for every $E_i \in \mathcal{E}$ and $H \in \mathcal{H}$ as*

$$\Pi(E_i|H) = \begin{cases} \frac{1}{i} & \text{if } E_i \wedge H \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The function Π is a coherent min*-conditional possibility as it can be extended as a* min*-conditional possibility on $\mathcal{B} \times \mathcal{H}$, where $\mathcal{B}$ is the field of finite-cofinite subsets of $\mathbb{N}$. For example, a possible extension is the function Π' defined for $H \in \mathcal{H}$ putting $\Pi'(E|H) = 1$ if E is cofinite, while if E is finite we set*

$$\Pi'(E|H) = \begin{cases} \frac{1}{\min\{i \,:\, i \in E \wedge H\}} & \text{if } E \wedge H \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Actually, Π' turns out to be a T-conditional possibility for every continuous t-norm T. Indeed, conditions (i) and (ii) are easily verified, while condition (iii) reduces to

$$\Pi'(E \wedge H_1) = T(\Pi'(E|H_1), \Pi'(H_1)),$$

for every $E \in \mathcal{B}$, which trivially holds since $\Pi'(H_1) = 1$ and $\Pi'(E \wedge H_1) = \Pi'(E|H_1)$ for every $E \in \mathcal{B}$.

We want to determine the coherent extension interval of the coherent min*-conditional possibility Π to the new event $H_1 = H_1|\mathbb{N}$. By previous discussion we know that 1 is the upper bound, thus we only need to compute the lower bound. Recalling that $\mathcal{E} \times \mathcal{H}$ is a countable set, for every $\{i_1, \dots, i_n\} \subseteq \mathbb{N}$ we can focus on the family $\mathcal{F} = \{E_{i_j}, E_{i_j}|H_1, : j = 1, \dots, n\}$. Indeed, by virtue of Remark 6 every finite subset of $\mathcal{F}$ gives rise to a larger extension interval, thus it can be ignored.*

Denote with $C_{i_j} = E_{i_j} \wedge H_1$ and $C'_{i_j} = E_{i_j} \wedge H_1^c$, $j = 1, \dots, n$, and $C_{i_{n+1}} = \bigwedge_{j=1}^n E^c_{i_j} \wedge H_1$ and $C'_{i_{n+1}} = \bigwedge_{j=1}^n E^c_{i_j} \wedge H^c_1$, the atoms generated by $\{E_{i_j}, H_1 : j = 1, \dots, n\}$, where only possible ones are considered.

The lower bound of the extension interval of $\Pi_{|\mathcal{F}}$ on H_1 is computed solving the following optimization problem under the system $\mathcal{S}^\Pi_{\mathcal{F}_0}$ [27], which has unknowns $x^0_{i_j}, x^0_{i_j}{}' \geq 0$ for atoms C_{i_j}, C'_{i_j}, $j = 1, \dots, n+1$, and results to be

$$\text{minimize} \left[\max_{j=1,\dots,n+1} \{x^0_{i_j}\} \right]$$

$$\mathcal{S}^\Pi_{\mathcal{F}0} : \begin{cases} \max\{x^0_{i_j}, x^0_{i_j}{}'\} = \frac{1}{i_j} \\ [j = 1, \dots, n] \\ \\ x^0_{i_j} = \min\left\{\frac{1}{i_j}, \max\limits_{j=1,\dots,n+1}\{x^0_{i_j}\}\right\} \\ [j = 1, \dots, n] \\ \max\limits_{j=1,\dots,n+1}\{x^0_{i_j}, x^0_{i_j}{}'\} = 1 \end{cases}$$

where equations of the second kind in which $C_{i_j} = \emptyset$ are neglected as well as unknowns corresponding to $C_{i_j} = \emptyset$ or $C'_{i_j} = \emptyset$.

The lower bound can be written as $m_{\{i_1,\dots,i_n\}} = \max\left\{\frac{1}{i_j} : j = 1, \dots, n, i_j \neq 1\right\}$.

Hence, the coherent min*-conditional possibility values* *for H_1 range in the closed interval*

$$\bigcap_{\{i_1,\dots,i_n\}\subseteq\mathbb{N}} [m_{\{i_1,\dots,i_n\}}, 1] = \left[\frac{1}{2}, 1\right].$$

3 Possibilistic likelihood functions and possibilistic priors on infinite partitions

Theorem 1 and 2 deal with coherence and extension in their most general form. Nevertheless, there are situations in which coherence is immediately implied by some conditions and the extension on a new conditional event is easily computed.

This is the case of Bayesian-like inference processes in which one considers a *prior possibility* $\pi(\cdot)$ on a partition $\{H_i\}_{i\in I}$ and a *possibilistic likelihood* $f(E|\cdot)$ on the set $\{E|H_i\}_{i\in I}$, where E is the *evidence* event. The aim is to evaluate the *posterior possibility* of the conditional events $\{H_i|E\}_{i\in I}$.

To accomplish this task it is fundamental to establish whether the two assessments π and f are coherent *per se* and moreover whether the global assessment $\{f,\pi\}$ is coherent.

A complete characterization of the coherence of previous assessments has been given for a finite $I = \{1,\dots,n\}$ in [9]. In this case, the coherence of $\{f,\pi\}$ allows to regard the global assessment as a $\Pi(\cdot|\cdot)$ on the set $\mathcal{G} = \{H_i, E|H_i\}_{i\in I}$ and to apply the following possibilistic counterpart of the *Bayes formula* (where we denote with Π also the posterior) for $i = 1,\dots,n$,

$$T\left(\Pi(H_i|E), \max_{j=1,\dots,n}\{T(\Pi(E|H_j),\Pi(H_j))\}\right) = \\ = T(\Pi(E|H_i),\Pi(H_i)). \quad (3)$$

Notice that, differently from the probabilistic case, depending on the particular t-norm T, the posterior possibility $\Pi(\cdot|E)$ could be non-unique on some H_i even requiring $\Pi(E) > 0$. In particular, if we consider $T = \min$ or a strict t-norm, Theorem 2 implies that each posterior $\Pi(H_i|E)$ lies in a (possibly degenerate) closed interval. Hence, in case of non-uniqueness, an arbitrary value in each interval can be chosen: the only constraint we have is that $\max_{i=1,\dots,n}\Pi(H_i|E) = 1$.

Example 2. *Consider the finite partition $\mathcal{L} = \{H_1,H_2,H_3\}$ together with the event E such that $E\wedge H_1 = \emptyset$. The following global assessment $\Pi(H_1) = 1$, $\Pi(H_2) = \Pi(H_3) = \frac{1}{3}$, $\Pi(E|H_1) = 0$, $\Pi(E|H_2) = \frac{1}{2}$ and $\Pi(E|H_3) = \frac{1}{3}$, is a coherent* min*-conditional possibility.*

In order to get the posterior (that we still denote with Π) we compute

$$\max_{j=1,2,3}\{\min\{\Pi(E|H_j),\Pi(H_j)\}\} = \frac{1}{3},$$

thus for $i = 1,2,3$ we need to solve

$$\min\left\{\Pi(H_i|E), \frac{1}{3}\right\} = \min\{\Pi(E|H_i),\Pi(H_i)\},$$

that implies $\Pi(H_1|E) = 0$, $\Pi(H_2|E), \Pi(H_3|E) \in \left[\frac{1}{3}, 1\right]$ such that $\max\{\Pi(H_2|E),\Pi(H_3|E)\} = 1$.

Our goal in this section is to generalize previous results to the case of an infinite index set I with $\operatorname{card} I \geq \operatorname{card}\mathbb{N}$.

Next theorem puts in evidence that every function defined on an infinite partition $\mathcal{L} = \{H_i\}_{i\in I}$ and ranging in $[0,1]$ (in particular the null function) is a coherent finitely maxitive possibility (i.e., it can be extended as a finitely maxitive possibility on $\langle\mathcal{L}\rangle$), and so, by Remark 1, a coherent T-conditional possibility, for any continuous t-norm T.

Theorem 3. *Let $\mathcal{L} = \{H_i\}_{i\in I}$ be a partition of Ω with $\operatorname{card} I \geq \operatorname{card}\mathbb{N}$. Then any function $\pi : \mathcal{L} \to [0,1]$ is a coherent T-conditional possibility (for every continuous t-norm T).*

Proof. We use condition *(ii)* of Theorem 1. Then for every $\{i_1,\dots,i_n\} \subseteq I$, take the set $\mathcal{F} = \{H_{i_j} : j = 1,\dots,n\}$ and denote $C_{i_j} = H_{i_j}$ for $j = 1,\dots,n$, and $C_{i_{n+1}} = \bigwedge_{j=1}^n H_{i_j}^c$, the atoms generated by $\mathcal{F}$.

Consider the sequence of systems $\mathcal{S}^{\Pi}_{\mathcal{F}\alpha}$ with $\mathcal{H}_{\mathcal{F}} = \{\Omega\}$. The first (and unique) system of the sequence has unknowns $x^0_{i_j} \geq 0$ for C_{i_j}, $j = 1,\dots,n+1$, and results to be

$$\mathcal{S}^{\Pi}_{\mathcal{F}0} : \begin{cases} x^0_{i_j} = \pi(H_{i_j}) & j = 1,\dots,n \\ \max_{j=1,\dots,n+1}\{x^0_{i_j}\} = 1. \end{cases}$$

System $\mathcal{S}^{\Pi}_{\mathcal{F}0}$ admits the solution $x^0_{i_j} = \pi(H_{i_j})$, for $j = 1,\dots,n$, and $x^0_{i_{n+1}} = 1$, and so π is coherent. $\square$

Let $\mathcal{L} = \{H_i\}_{i\in I}$ be an arbitrary partition of Ω and E an arbitrary event, in the following we call *likelihood function* any function $f : \{E\}\times\mathcal{L} \to [0,1]$ defined as:

$$f(E|H_i) = \begin{cases} 0 \text{ when } E\wedge H_i = \emptyset, \\ 1 \text{ when } H_i \subseteq E, \\ \text{a value } \gamma_i \in [0,1] \text{ otherwise.} \end{cases} \quad (4)$$

We underline that for the values γ_i's the only constraint is to be between 0 and 1.

Theorem 4. *Let $\mathcal{L} = \{H_i\}_{i\in I}$ be a partition of Ω with $\operatorname{card} I \geq \operatorname{card}\mathbb{N}$ and E an arbitrary event. For a likelihood function $f : \{E\}\times\mathcal{L} \to [0,1]$, defined by (4), the following statements hold:*

(i) *f is a coherent conditional probability;*

(ii) *f is a coherent T-conditional possibility (for every continuous t-norm T).*

Proof. In [9] this theorem has been proved for a finite partition $\mathcal{L}$, we prove it for the infinite case. Condition *(i)* follows by Proposition 1 in [8] and Theorem 4 in [11]. To prove *(ii)*, by condition *(ii)* of Theorem 1, for every $\{i_1, \dots, i_n\} \subseteq I$, take the set $\mathcal{F} = \{E|H_{i_j} : j = 1, \dots, n\}$ and denote $C_{i_j} = E \wedge H_{i_j}$ and $C'_{i_j} = E^c \wedge H_{i_j}$ for $j = 1, \dots, n$, and $C_{i_{n+1}} = E \wedge \bigwedge_{j=1}^n H^c_{i_j}$ and $C'_{i_{n+1}} = E^c \wedge \bigwedge_{j=1}^n H^c_{i_j}$, the atoms generated by $\{E, H_{i_j} : j = 1, \dots, n\}$, where only possible ones are considered.

Consider the sequence of systems $\mathcal{S}^{\Pi}_{\mathcal{F}\alpha}$ with $\mathcal{H}_{\mathcal{F}}$ equal to the additive set generated by the H_{i_j}'s. The first (and unique) system of the sequence has unknowns $x^0_{i_j}, {x^0_{i_j}}' \geq 0$ for C_{i_j}, C'_{i_j}, $j = 1, \dots, n$, and results to be

$$\mathcal{S}^{\Pi}_{\mathcal{F}0} : \begin{cases} x^0_{i_j} = T\left(f(E|H_{i_j}), \max\{x^0_{i_j}, {x^0_{i_j}}'\}\right) \\ [j = 1, \dots, n] \\ \\ \max\limits_{j=1,\dots,n} \{x^0_{i_j}, {x^0_{i_j}}'\} = 1 \end{cases}$$

where equations in which $C_{i_j} = \emptyset$ are neglected as well as unknowns corresponding to $C_{i_j} = \emptyset$ or $C'_{i_j} = \emptyset$. A solution for $\mathcal{S}^{\Pi}_{\mathcal{F}0}$ is $x^0_{i_j} = f(E|H_{i_j})$ and ${x^0_{i_j}}' = 1$ for $j = 1, \dots, n$, implying that f is coherent. □

Previous theorem highlights that no significant property characterizes a likelihood function (defined by (4)) regarded either as coherent conditional probability or as coherent T-conditional possibility.

Remark 7. *We notice that Theorem 4 is related to a function defined only on a set of events $\{E\} \times \mathcal{L}$, (the conditioned event E is only one). Obviously, if we have a family of likelihood functions $\{f_j : j \in J\}$ each defined on $\{E_j\} \times \mathcal{L}$, where $\mathcal{E} = \{E_j\}_{j \in J}$ is an arbitrary set, the assessment could be non-globally coherent. In particular if $\mathcal{E}$ is a finite partition we must take into account additivity in the probabilistic case and maxitivity in the possibilistic case, as the following Theorem 5 shows.*

Theorem 5. *Let $\mathcal{E} = \{E_j\}_{j=1,\dots,m}$ and $\mathcal{L} = \{H_i\}_{i \in I}$ be two partitions and let $\mathcal{F}$ be a (finite) class $\{f_j : j = 1, \dots, m\}$ of likelihood functions, where each f_j is defined by (4) on $\{E_j\} \times \mathcal{L}$, for $j = 1, \dots, m$. Then the following statements hold:*

(i) *the global assessment $\mathcal{F}$ is a coherent conditional probability if and only if $\sum_{j=1}^{m} f_j(E_j|H_i) = 1$ for every H_i;*

(ii) *the global assessment $\mathcal{F}$ is a coherent T-conditional possibility (for every continuous t-norm T) if and only if $\max\limits_{j=1,\dots,m} f_j(E_j|H_i) = 1$ for every H_i.*

Proof. In [9] this theorem has been proved for a finite partition $\mathcal{L}$, we prove it for the infinite case. Condition *(i)* follows by Theorem 4 in [11]. Condition *(ii)* follows by Theorem 1 on the same line of the proof of Theorem 4. □

Next theorem focuses on a likelihood function taking into account also a probabilistic or possibilistic prior.

Theorem 6. *Let $\mathcal{L} = \{H_i\}_{i \in I}$ be a partition of Ω with $\operatorname{card} I \geq \operatorname{card} \mathbb{N}$ and E an arbitrary event. Consider a likelihood function $f : \{E\} \times \mathcal{L} \to [0, 1]$, defined by (4), a coherent probability assessment $p : \mathcal{L} \to [0, 1]$ and a coherent possibility assessment $\pi : \mathcal{L} \to [0, 1]$. The following statements hold:*

(i) *the global assessment $\{f, p\}$ is a coherent conditional probability;*

(ii) *the global assessment $\{f, \pi\}$ is a coherent T-conditional possibility (for every continuous t-norm T).*

Proof. In [9] this theorem has been proved for a finite partition $\mathcal{L}$, we prove it for the infinite case. Condition *(i)* follows by Proposition 2 in [8] and Theorem 4 in [11] (see also [28, 32]). Condition *(ii)* follows by Theorem 1 in analogy to the proof of Theorem 4, and taking into account Remark 5. □

Example 3. *Consider $\mathbb{N}$ as universe and take the partition $\mathcal{L} = \{H_i = \{2i - 1, 2i\}\}_{i \in \mathbb{N}}$, together with $E = \{2i : i \in \mathbb{N}\}$. Consider the assessments $f(E|H_i) = \frac{1}{i}$, $p(H_i) = \pi(H_i) = 0$ for $i \in \mathbb{N}$. We have that $f(E|\cdot)$ verifies condition (4), moreover $p(\cdot)$ and $\pi(\cdot)$ are, respectively, a coherent probability and a coherent possibility. This implies $\{f, p\}$ and $\{f, \pi\}$ are, respectively, a coherent conditional probability and a coherent T-conditional possibility (for every continuous T-norm).*

4 Complete disintegrability and complete conglomerability

In this section we consider a T-conditional possibility Π on $\mathcal{B} \times \mathcal{H}$, with $\mathcal{H}$ containing Ω and a partition $\mathcal{L} = \{H_i\}_{i \in I}$, where I is arbitrary. Moreover, we say that an event $E \in \mathcal{B}$ is *logically independent* of the elements of $\mathcal{L}$ if $\emptyset \neq E \wedge H_i \neq H_i$, for $i \in I$.

Definition 4. *A T-conditional possibility Π on $\mathcal{B}\times\mathcal{H}$ is* **completely maxitive on** $\mathcal{L}$ *if it holds*

$$\sup_{i\in I}\Pi(H_i) = 1. \tag{5}$$

Definition 5. *Given an event $E \in \mathcal{B}$, and a T-conditional possibility Π on $\mathcal{B}\times\mathcal{H}$, we say that Π is* **completely $\mathcal{L}$-disintegrable on** E *if it holds*

$$\Pi(E) = \sup_{i\in I} T(\Pi(E|H_i), \Pi(H_i)). \tag{6}$$

We introduce now a notion of conglomerability analogous the one introduced by de Finetti [17, 18, 19] (see also [29, 7, 30, 31, 1]), involving only events. We recall that in probability theory a stronger notion of conglomerability involving linear spaces of bounded random variables is present (see for instance [21, 28, 4]).

Definition 6. *Given an event $E \in \mathcal{B}$, and a T-conditional possibility Π on $\mathcal{B}\times\mathcal{H}$, we say that Π is* **completely $\mathcal{L}$-conglomerative on** E *if it holds*

$$\inf_{i\in I}\Pi(E|H_i) \leq \Pi(E) \leq \sup_{i\in I}\Pi(E|H_i). \tag{7}$$

Remark 8. *Definitions 5 and 6 actually involve only a family $\mathcal{G} = \{E, H_i, E|H_i\}_{i\in I}$ contained in $\mathcal{B}\times\mathcal{H}$, so they can be given for a coherent T-conditional possibility assessment on $\mathcal{G}$, if we are interested only on complete $\mathcal{L}$-conglomerability or complete $\mathcal{L}$-disintegrability on E (for instance in Bayesian-like updating). In fact, these properties are satisfied (for the given E and $\mathcal{L}$) by all the possible extensions on $\mathcal{B}\times\mathcal{H}$. Nevertheless, as discussed in the following, the above properties required only for one event E are not particularly meaningful, so we use a Π on $\mathcal{B}\times\mathcal{H}$ to enforce the properties to all the events of $\mathcal{B}$.*

In the case the partition $\mathcal{L}$ is finite, it is readily verified that complete maxitivity on $\mathcal{L}$ collapses into finite maxitivity and complete $\mathcal{L}$-disintegrability and complete $\mathcal{L}$-conglomerability always hold for every $E \in \mathcal{B}$, as simple implications of Definition 1. Nevertheless, previous properties could not be verified when the partition is infinite. In particular, in analogy with finitely additive conditional probability [18, 29], there can exist events $E \in \mathcal{B}$ on which Π is completely $\mathcal{L}$-disintegrable but not completely $\mathcal{L}$-conglomerative and vice versa, as shown in next example.

Example 4. *Let T be a continuous t-norm and consider the countable set $\mathcal{G} = \{E, H_i, E|H_i\}_{i\in\mathbb{N}}$ with E logically independent of the elements of the partition $\mathcal{L} = \{H_i\}_{i\in\mathbb{N}}$. Recall that the coherence of an assessment on $\mathcal{G}$ implies its extendability on $\mathcal{B}\times\mathcal{H}$, where $\mathcal{B} = \langle\{E\}\cup\mathcal{L}\rangle$ and $\mathcal{H}$ is the additive set generated by $\mathcal{L}$.*

The coherent T-conditional possibility assessment $\Pi(E) = \frac{1}{2}$, $\Pi(E|H_i) = \frac{1}{i}$ and $\Pi(H_i) = 0$ for $i \in \mathbb{N}$ is completely $\mathcal{L}$-conglomerative on E, but not completely $\mathcal{L}$-disintegrable on E. In fact, we have $\Pi(E) = \frac{1}{2} \neq 0 = \sup_{i\in I} T(\Pi(E|H_i), \Pi(H_i))$

On the other hand, the coherent assessment $\Pi(E) = \Pi(H_i) = 0$ and $\Pi(E|H_i) = \frac{1}{2}$ for $i \in \mathbb{N}$ is completely $\mathcal{L}$-disintegrable on E, but it is not completely $\mathcal{L}$-conglomerative on E, since we have $\Pi(E) = 0 < \frac{1}{2} = \inf_{i\in I}\Pi(E|H_i)$.

Previous claim suggests to give a definition of complete $\mathcal{L}$-disintegrability and complete $\mathcal{L}$-conglomerability which is not dependent on the event E.

Definition 7. *A T-conditional possibility Π on $\mathcal{B}\times\mathcal{H}$ is* **completely $\mathcal{L}$-disintegrable** *if it is completely $\mathcal{L}$-disintegrable on E, for every $E \in \mathcal{B}$.*

Definition 8. *A T-conditional possibility Π on $\mathcal{B}\times\mathcal{H}$ is* **completely $\mathcal{L}$-conglomerative** *if it is completely $\mathcal{L}$-conglomerative on E, for every $E \in \mathcal{B}$.*

Let us note that the notion of conglomerability given in previous definition differs from the ones proposed for coherent lower and upper previsions (see for instance [34, 16, 26]). The difference is essentially due to the different concepts of conditioning adopted (see Remark 2).

Remark 9. *Suppose to have a possibilistic prior π on a partition $\mathcal{L}$ and two likelihood functions f_j on $\{E_j\}\times\mathcal{L}$, with $E_j \in \mathcal{B}$, ($j = 1, 2$), such that each $\{f_j, \pi\}$ admits a completely $\mathcal{L}$-conglomerative extension on $\mathcal{B}\times\mathcal{H}$. Even in the case $\{f_1, f_2, \pi\}$ is globally coherent there could not exist a completely $\mathcal{L}$-conglomerative extension on $\mathcal{B}\times\mathcal{H}$ (similarly for complete $\mathcal{L}$-disintegrability). Previous discussion generalizes to a larger class of likelihood functions.*

It is well-known that, in the probabilistic framework (see for instance [18, 21, 29]), for a countable I, $\mathcal{L}$-disintegrability and σ-additivity on $\mathcal{L}$ are equivalent. Nevertheless, since in the case of probability the equivalence is implied by the subtractive property, the same equivalence does not hold in the case of possibility, as shown by next example.

Example 5. *Let T be a continuous t-norm and I an index set s.t.* $\operatorname{card} I \geq \operatorname{card}\mathbb{N}$. *Consider the set $\mathcal{G} = \{E, H_i, E|H_i\}_{i\in I}$, where the H_i's form a partition $\mathcal{L}$ of Ω and E is logically independent of the H_i's.*

The assessment $\Pi(E) = \Pi(H_i) = 1$ and $\Pi(E|H_i) = 0$ for $i \in I$, is a coherent T-conditional possibility.

We have that Π is completely maxitive on the partition $\mathcal{L}$ since $\sup_{i\in I}\Pi(H_i) = 1$, while it is not completely $\mathcal{L}$-disintegrable on E since $\Pi(E) = 1 \neq 0 =$

$\sup_{i\in I} T(\Pi(E|H_i), \Pi(H_i))$.

In the possibilistic setting, complete maxitivity on $\mathcal{L}$ is only a necessary condition for complete $\mathcal{L}$-disintegrability.

Proposition 1. *If a coherent T-conditional possibility Π on $\mathcal{B}\times\mathcal{H}$ is completely $\mathcal{L}$-disintegrable, then it is completely maxitive on $\mathcal{L}$.*

Proof. It holds

$$1 = \Pi(\Omega) = \sup_{i\in I} T(\Pi(\Omega|H_i), \Pi(H_i)) = \sup_{i\in I} \Pi(H_i).$$

□

We notice that if Π is not completely maxitive on $\mathcal{L}$ then, if there exists an $E \in \mathcal{B}$ such that Π is completely $\mathcal{L}$-disintegrable on E then Π is not completely $\mathcal{L}$-disintegrable on E^c.

Next theorem shows that, analogously to the probabilistic case [19], complete $\mathcal{L}$-disintegrability implies the complete $\mathcal{L}$-conglomerative property.

Theorem 7. *If a T-conditional possibility Π on $\mathcal{B}\times\mathcal{H}$ is completely $\mathcal{L}$-disintegrable, then it is completely $\mathcal{L}$-conglomerative.*

Proof. For every $E \in \mathcal{B}$, complete $\mathcal{L}$-disintegrability implies that

$$\Pi(E) = \sup_{i\in I} T(\Pi(E|H_i), \Pi(H_i)) \le \sup_{i\in I} \Pi(E|H_i),$$

moreover, setting $\kappa = \inf_{i\in I} \Pi(E|H_i)$ and recalling Proposition 1 and that any left-continuous t-norm commutes with the supremum, we get

$$\begin{aligned} \Pi(E) &= \sup_{i\in I} T(\Pi(E|H_i), \Pi(H_i)) \\ &\ge \sup_{i\in I} T(\kappa, \Pi(H_i)) = T\left(\kappa, \sup_{i\in I} \Pi(H_i)\right) = \kappa. \end{aligned}$$

□

Nevertheless, as it is shown in [29] for probability theory in the case of a countable partition, complete $\mathcal{L}$-disintegrability and complete $\mathcal{L}$-conglomerability are not equivalent. The next example in fact shows that complete $\mathcal{L}$-disintegrability is just a sufficient condition for the complete $\mathcal{L}$-conglomerative property.

Example 6. *Take $\mathbb{N}$ as universe, let $\mathcal{B}$ be the field of finite-cofinite subsets of $\mathbb{N}$ and $\mathcal{L} = \{H_i = \{i\}\}_{i\in\mathbb{N}}$. Consider on $\mathcal{B}\times\mathcal{B}^0$ the function Π defined for any $E|H \in \mathcal{B}\times\mathcal{B}^0$ putting if H is cofinite*

$$\Pi(E|H) = \begin{cases} 0 & \text{if } E\wedge H \text{ is finite,} \\ 1 & \text{otherwise,} \end{cases}$$

while if H is finite

$$\Pi(E|H) = \begin{cases} 0 & \text{if } E\wedge H = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

First we show that Π is a full T-conditional possibility on $\mathcal{B}$ for any continuous t-norm T. For this, it is sufficient to show that axiom (iii) of Definition 1 is satisfied, since axioms (i) and (ii) are easily seen to be verified. At this aim, for any $H, E\wedge H \in \mathcal{B}^0$ and $E, F \in \mathcal{B}$ we consider the following cases.

(Case 1). If $E\wedge H$ and H are cofinite then we have $\Pi(E|H) = 1$, thus axiom (iii) is verified both when $E\wedge F\wedge H$ is cofinite (in this case we have $\Pi(E\wedge F|H) = \Pi(F|E\wedge H) = 1$) and when $E\wedge F\wedge H$ is finite (in this case we have $\Pi(E\wedge F|H) = \Pi(F|E\wedge H) = 0$).

(Case 2). If $E\wedge H$ is finite and H is cofinite then we have $\Pi(E|H) = 0$, thus axiom (iii) is verified for every value of $\Pi(F|E\wedge H)$, since $E\wedge F\wedge H$ is finite and so we have $\Pi(E\wedge F|H) = 0$.

(Case 3). If $E\wedge H$ and H are finite then we have $\Pi(E|H) = 1$, thus axiom (iii) is verified both when $E\wedge F\wedge H \ne \emptyset$ (in this case we have $\Pi(E\wedge F|H) = \Pi(F|E\wedge H) = 1$) and when $E\wedge F\wedge H = \emptyset$ (in this case we have $\Pi(E\wedge F|H) = \Pi(F|E\wedge H) = 0$).

It is easily seen that Π is not completely maxitive on $\mathcal{L}$, since

$$\Pi(\mathbb{N}) = 1 > 0 = \sup_{i\in\mathbb{N}} \Pi(H_i),$$

thus by virtue of Proposition 1, Π is not completely $\mathcal{L}$-disintegrable. On the contrary, we have that Π is completely $\mathcal{L}$-conglomerative. Indeed, if E is cofinite we have $\Pi(E) = 1 \ge \inf_{i\in\mathbb{N}} \Pi(E|H_i)$, and there must exist $j \in \mathbb{N}$ such that $E\wedge H_j \ne \emptyset$, thus $\sup_{i\in\mathbb{N}} \Pi(E|H_i) = 1$. Moreover, if E is finite we have $\Pi(E) = 0 \le \sup_{i\in\mathbb{N}} \Pi(E|H_i)$, and there must exist $j \in \mathbb{N}$ such that $E\wedge H_j = \emptyset$, thus $\inf_{i\in\mathbb{N}} \Pi(E|H_i) = 0$.

Since complete $\mathcal{L}$-disintegrability and complete $\mathcal{L}$-conglomerability refer to a partition $\mathcal{L} \subset \mathcal{H}$, it is natural to ask if their validity w.r.t. an infinite $\mathcal{L}$ implies the validity w.r.t. any other infinite partition $\mathcal{L}' \subset \mathcal{H}$. In next example, inspired to the well-known Lévy's paradox [19, 30, 31, 7], we show that it is not the case.

Example 7. *Take $\mathbb{N}^2$ as universe, let $\mathcal{B}$ be the power set of $\mathbb{N}^2$ and take the two partitions $\mathcal{L}_1 = \{H_i = \{i\}\times\mathbb{N}\}_{i\in\mathbb{N}}$ and $\mathcal{L}_2 = \{K_i = \mathbb{N}\times\{i\}\}_{i\in\mathbb{N}}$. Consider on $\mathcal{B}\times(\mathcal{L}_1\cup\mathcal{L}_2)$ the function Π defined for any $E|H \in \mathcal{B}\times(\mathcal{L}_1\cup\mathcal{L}_2)$ putting*

$$\Pi(E|H) = \begin{cases} 0 & \text{if } E\wedge H \text{ is finite,} \\ 1 & \text{otherwise.} \end{cases}$$

It is possible to show that the assessment Π *is a coherent* T*-conditional possibility for any continuous* t*-norm* T.

The coherence of Π *implies its extendability to* $\mathcal{B}\times\mathcal{H}$*, where* $\mathcal{H}$ *is the additive set generated by* $\mathcal{L}_1 \cup \mathcal{L}_2$*. In particular, taking* $E = \{(i,j) \in \mathbb{N}^2 \ : \ i \geq j\}$ *we have* $\Pi(E|H_i) = \Pi(E^c|K_i) = 0$*, for any* $i \in \mathbb{N}$*, which implies that no extension* Π' *can be simultaneously completely* $\mathcal{L}_1$*-conglomerative and completely* $\mathcal{L}_2$*-conglomerative.*

Finally, by virtue of Theorem 7 it follows that no extension Π' *can be simultaneously completely* $\mathcal{L}_1$*-disintegrable and completely* $\mathcal{L}_2$*-disintegrable.*

Complete $\mathcal{L}$-disintegrability and complete $\mathcal{L}$-conglomerability are particularly relevant in the context of Bayesian-like inference processes since they constrain the set of coherent values for the posterior possibility. Anyway, when they are not satisfied, one needs to go back to the general enlargement procedure in which the posterior values are determined by Theorem 2.

For this, we are interested in the coherent extensions Π' on $\mathcal{G}\cup\{E\}$ of a coherent T-conditional possibility Π assessed on a family $\mathcal{G} = \{H_i, E|H_i\}_{i\in I}$, $\operatorname{card} I \geq \operatorname{card}\mathbb{N}$, where the set $\mathcal{L} = \{H_i\}_{i\in I}$ is a partition of Ω and E is an arbitrary event. Let us stress that Π is nothing else than the global assessment corresponding to a likelihood f and a possibilistic prior π (coherent by Theorem 6).

Next theorem characterizes the set of coherent values for the possibility $\Pi'(E)$ in the case E is logically independent of the H_i's and T is the minimum or a strict t-norm. Notice that if $H_i \subseteq E$ for every $i \in I$, then it must be $\Pi(E|H_i) = 1$ for every $i \in I$ and so $\Pi'(E) = 1$; similarly, if $H_i \wedge E = \emptyset$ for every $i \in I$, then it must be $\Pi(E|H_i) = 0$ for every $i \in I$ and so $\Pi'(E) = 0$. Thus in this two trivial situations complete $\mathcal{L}$-conglomerability on E holds compulsorily.

Theorem 8. *Let* Π *be a coherent* T*-conditional possibility on* $\mathcal{G}$ *(with* $T = \min$ *or strict) such that for* $i \in I$ *it is* $\emptyset \neq E \wedge H_i \neq H_i$, $\Pi(E|H_i) = \pi_i$ *and* $\Pi(H_i) = \pi_i'$*, with* $\operatorname{card} I \geq \operatorname{card}\mathbb{N}$*. Then the set of coherent values for* $\Pi'(E)$ *is*

$$\bigcap_{\{i_1,\ldots,i_n\}\subseteq I} \left[M_{\{i_1,\ldots,i_n\}}, 1\right], \qquad (8)$$

where $M_{\{i_1,\ldots,i_n\}} = \max\limits_{j=1,\ldots,n} T(\pi_{i_j}, \pi'_{i_j})$.

Proof. By Theorem 2 the coherent values for $\Pi'(E)$ are a closed interval $[\pi_*, \pi^*]$, that is obtained as the intersection of all the intervals $[\pi_{\mathcal{F}*}, \pi_{\mathcal{F}}^*]$ expressing the coherent extensions of $\Pi_{|\mathcal{F}}$ on E, for any finite subfamily $\mathcal{F} \subseteq \mathcal{G}$.

Thus, for every $\{i_1,\ldots,i_n\} \subseteq I$ take the set $\mathcal{F} = \{H_{i_j}, E|H_{i_j} : j = 1,\ldots,n\}$. Notice that by Remark 6 every finite subset of $\mathcal{F}$ gives rise to a larger extension interval than the one induced by $\mathcal{F}$ and thus can be ignored. Denote with $C_{i_j} = E \wedge H_{i_j}$ and $C'_{i_j} = E^c \wedge H_{i_j}$, $j = 1,\ldots,n$, and $C_{i_{n+1}} = E \wedge \bigwedge_{j=1}^n H_{i_j}^c$ and $C'_{i_{n+1}} = E^c \wedge \bigwedge_{j=1}^n H_{i_j}^c$, the atoms generated by $\{E, H_{i_j} : j = 1,\ldots,n\}$.

The endpoints of the extension interval of $\Pi_{|\mathcal{F}}$ on E are computed solving the following two optimization problems under the system $\mathcal{S}^{\Pi}_{\mathcal{F}_0}$, which has unknowns $x^0_{i_j}, {x^0_{i_j}}' \geq 0$ for atoms C_{i_j}, C'_{i_j}, $j = 1,\ldots,n+1$, and result to be

$$\text{minimize} \Big/ \text{maximize} \left[\max_{j=1,\ldots,n+1}\{x^0_{i_j}\}\right]$$

$$\mathcal{S}^{\Pi}_{\mathcal{F}0} : \begin{cases} \max\{x^0_{i_j}, {x^0_{i_j}}'\} = \pi'_{i_j} & j = 1,\ldots,n \\ x^0_{i_j} = T\left(\pi_{i_j}, \max\{x^0_{i_j}, {x^0_{i_j}}'\}\right) & j = 1,\ldots,n \\ \max\limits_{j=1,\ldots,n+1}\{x^0_{i_j}, {x^0_{i_j}}'\} = 1 \end{cases}$$

for which any solution is such that $x^0_{i_j} = T(\pi_{i_j}, \pi'_{i_j})$, for $j = 1,\ldots,n$, thus the possibility of E is determined by the value assigned to $x^0_{i_{n+1}}$ which is only asked to belong to $[0,1]$. This implies the extension of $\Pi_{|\mathcal{F}}$ on E ranges in $\left[M_{\{i_1,\ldots,i_n\}}, 1\right]$ with $M_{\{i_1,\ldots,i_n\}} = \max\limits_{j=1,\ldots,n} T(\pi_{i_j}, \pi'_{i_j})$, and the conclusion follows. □

In particular, previous theorem implies that if $\Pi(E|H_i) = \pi$ for $i \in I$, then the extension Π' on $\mathcal{G}\cup\{E\}$ of every coherent T-conditional possibility Π on $\mathcal{G}$ is generally not completely $\mathcal{L}$-conglomerative on E if $\pi < 1$, since the value $\Pi'(E) = 1$ is always coherent. Theorem 8 also implies the coherence of the posterior (that we still denote with Π) defined as:

$$\Pi(H_i|E) = T(\Pi(E|H_i), \Pi(H_i)) \quad \text{for } i \in I. \qquad (9)$$

5 Conclusions

In probability theory, in particular in modern Bayesian analysis, concepts of conglomerability and disintegrability have been deeply studied, especially with respect to finitely additive probability, where many famous examples of nonconglomerative conditional probability assessments are proposed. We studied the analogous concepts in possibility theory, starting from the definition of finitely maxitive T-conditional possibility, with T any continuous t-norm. We put in evidence analogies and differences between the two frameworks.

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