An approach to uncertainty in probabilistic assignments with application to vibro-acoustic problems

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Abstract

In this paper a novel imprecise probability description is applied to vibro-acoustic problems in engineering. Frequently little data is available concerning the variability of the key input parameters required for a predictive analysis. This has led to widespread use of several uncertainty descriptions. The hybrid Finite Element/Statistical Energy Analysis (FE/SEA) approach to the analysis of vibro-acoustic systems is based on subdividing a system into: (i) SEA components which incorporate a non-parametric model of uncertainty and (ii) FE components with parametric uncertainty. This approach, combined with the Laplace asymptotic method, allows the evaluation of the failure probability. A novel strategy for establishing bounds on the failure probability when an imprecise probability model (based on expressing the probability density function of a random variable in the form of a maximum entropy distribution with bounded parameters) is employed is presented. The approach is illustrated by application to a built-up plate system.

Keywords. Uncertainties in probabilistic assignments, hybrid FE/SEA method, reliability analysis, parametric and non-parametric uncertainty models, maximum entropy distribution, vibro-acoustic analysis.

1 Introduction

In engineering problems it is frequently the case that little data is available concerning the variability of the key input parameters (geometry, material properties, and boundary conditions) required for a predictive analysis, and yet an engineering assessment of a design must nonetheless be performed. This topic has been the subject of much recent research, and various analytical and computational approaches have been proposed (for example, [1-9]). Such methods require some description of the underlying uncertainties (for example, uncertainties in material properties, loading conditions, and fabrication details) which could be probabilistic (parametric [1-4], non-parametric [5-7] or a combination of both [8,9]) or non-probabilistic [1,4].

Reliability methods aim to estimate the probability that design targets will be met [10,11]; this probability is referred to as the reliability of the system. These methods are often based on a parametric probabilistic description of the uncertain parameters of the system and rely on the assumption that the statistical distributions (i.e. probability density function (pdf)) of these parameters are precisely known [12]. The parametric probabilistic description requires a large amount of empirical data if the pdf is constructed using a frequentist view. Alternatively the pdf may be interpreted as a statement of belief based on expert opinion, as in the subjective approach to probability theory [13]. The more common frequentist approach is concerned with the outcome of experiments performed (hypothetically or in reality) on large ensembles of systems; these ensembles may either be real (for example cars from a production line), or virtual but realizable in principle (such as an ensemble of manufactured satellites, when only one satellite may actually be built). In contrast, with the subjective approach, no ensemble is necessarily involved. The frequentist and subjective views can be roughly aligned to the notions of aleatory and epistemic uncertainty; aleatory uncertainty is an irreducible uncertainty associated with an inherent variability of the properties of the system, while epistemic uncertainty is reducible, being associated with a lack of knowledge of the analyst with respect to the system's properties which are fixed [4]. Clearly, the interpretation employed for defining the pdf of the uncertain parameters will affect the interpretation of the results obtained with a predictive analysis.

In practice, only a limited amount of data may be available and therefore it is often difficult to identify the form of the distribution of the random variable and/or the parameters of the distribution. Moreover, the analyst may have uncertainties in belief, meaning that the specified pdf is itself subject to doubt. Using a pdf which differs from the actual one can significantly affect the prediction of the system performance with respect to safety, quality, design or cost constraints [12,14]. One way around this difficulty is to employ imprecise probability descriptions in the reliability assessment in order to establish bounds on the failure probability (that is the probability that the response exceeds a critical level). These bounds allow: (i) the evaluation of the sensitivity of the system response to the uncertainty of the system parameters; (ii) the identification of the worst case scenario (the highest failure probability expected). Many reliability approaches which includes imprecise probability descriptions have been developed in the past years, among which there are: (i) First Order Reliability Method (FORM) [10] approaches which employ pdfs with one [15] or two [16] bounded parameters (mean, variance or another distribution parameter), [15,16]; (ii) Dempster-Shafer theory (DST) [17,18] and P-box models of imprecise probabilities [19-21] applied to reliability analysis [22-25]; (iii) reliability analysis with random sets [26,27]; (iv) reliability assessment by means of Fuzzy Probabilities [4,28]; (v) Reliability models which account for the lack of information about the independence of the stress and strength, and about the parameters of each pdf [29]; (vi) reliability models based on imprecise Bayesian inference models [30]; (vii) Interval importance sampling methods combined with specified pdf with bounded parameters [31]. However, the application of these approaches is often limited to simple models, mainly because of the computational burden associated to the propagation of the imprecise probability description.

In automotive and aerospace industries there are design requirements to ensure vibro-acoustic performance is met. Vibro-acoustic problems usually involve a very broad frequency range due to the broadband nature of the loadings acting on the system. Broadly speaking three frequency ranges can be identified: low-, mid- and high-frequency ranges. In the low-frequency range the length scale of deformation of the system components is relatively long with respect to their overall dimension so that: (i) few degrees of freedom are required to model their dynamic behavior; (ii) the system response is insensitive to small changes in the system properties. The Finite Element method (FE) [32] is a well-established deterministic technique for acoustics and vibration analysis in the low-frequency range. In the high-frequency range, instead, the length scale of deformation is comparable to small manufacturing imperfections producing high sensitivity to uncertainty and requiring a large number of degrees of freedom for capturing the components' dynamic behavior. An alternative to FE is to employ Statistical Energy Analysis (SEA) [6,33], a probabilistic technique which was developed specifically to deal with high frequency vibration. In SEA the system is modeled as an assembly of subsystems, whose response is described by their vibrational energy (defined as twice the time-averaged kinetic energy). The number of degrees of freedom employed is drastically reduced compared to the FE approach, since a single degree of freedom SEA subsystem might replace thousand of finite element nodes. The interaction between the SEA subsystems is described using the principle of conservation of energy flow, and this leads to a set of equations that can be solved to yield the subsystem energies. This method can predict both the ensemble average vibrational energy levels [33] (averaged across an ensemble of nominally identical structures) and the ensemble variance of the energy levels [6]. The application of this approach is limited to high frequency because of its underlying assumptions (i.e. each structural component is sufficiently random and that the coupling between subsystems is sufficiently weak [6]). Between the respective ranges of validity of FE and SEA there is a *mid-frequency* region and much research effort has been directed at the development of efficient analytical methods that can be applied in this range. One such method is the hybrid FE/SEA method [7,34]. This approach is based on subdividing a system into SEA components (which incorporate a non-parametric probabilistic model of uncertainty), and deterministic FE components. This partition leads to a large reduction of the number of degrees of freedom employed in the model and a large gain in numerical efficiency. Moreover the method enables the prediction of the mean and variance of the response (such as the energy response of a SEA subsystem or the mean squared amplitude of the finite element degrees of freedom) over a collection of systems with random SEA subsystems properties [7,34] without employing Monte Carlo simulations. The hybrid FE/SEA method has been recently generalized by introducing parametric uncertainty into the FE components [8] in order to provide an enhanced description of those components which may contain a degree of randomness, but cannot be appropriately modeled as SEA subsystems. The vibro-acoustic performance of a complex system in a broad frequency range can be established by applying the hybrid FE/SEA method in combination with the Laplace's method (hybrid FE/SEA + Laplace) [35]. With this approach both parametric and non-parametric probabilistic uncertainty models are employed and the failure probability over the combined ensembles of uncertainty can be assessed. This approach is enhanced in this paper by considering a system with uncertain properties modeled with non-parametric, parametric and also imprecise parametric probabilistic descriptions in order to account for those input parameters of the FE components which are imprecisely known. In particular, the hybrid FE/SEA + Laplace is extended in this paper by employing a recently developed model of imprecise probability [36] in order to establish bounds on the failure probability. The imprecise model employed is based on expressing the probability density function of a random variable in the form of a maximum entropy distribution with bounded parameters [36]. This parametric probabilistic uncertainty model will be described in more details in Section 2. The hybrid FE/SEA + Laplace approach will be summarized in Section 3. In Section 4 an efficient approach for establishing bounds on the failure probability is presented. The method is illustrated by application to a built-up plate system in Section 5.

2 Probability Density Function with Bounded Parameters

In this Section a recently developed parametric model of uncertainty which admits uncertainty in the probabilistic assignments is described [36]. This uncertainty model requires as input bounded statistical expectations of specified functions of the random variable and it can be used to describe both aleatory and epistemic uncertainties. The uncertainty model is briefly described in Subsection 2.1. In Subsection 2.2, a procedure for treating the bounded statistical expectations is summarised.

2.1 Basic Concepts

The model of uncertainty is based on considering that the pdf of a random variable x itself is subject to doubt. The pdf is expressed as the exponential of a series expansion, but the parameters within this model, the so-called basic variables, are allowed to have bounded description [36]:

$$p(x|\mathbf{a}\in \mathrm{S}) = \exp\left[\sum_{j=1}^{n} a_j f_j(x)\right]. \tag{1}$$

Eq. (1) represents a family of distributions defined over the set of basic variables $\mathbf{a}$ (which has entries a_j with $j = 2,3...,n$) that lie within an admissible region S. A "basic variable" is defined here as one which can have any possible pdf within certain bounds, including the extreme case of a delta function at any point between the bounds. If a parameter is not "basic", then its pdf can be expressed in terms of the basic parameters, and thus only this type of parameter is considered in what follows. The admissible region S can be an interval, a convex region, etc. The term $f_j(x)$ is a specified function of the uncertain variable, such that $f_1(x) = 1$. The coefficient a_1 is dependent on the bounded basic variables a_j and it is chosen to satisfy the normalisation condition.
Eq. (1) describes a single distribution when the basic variables have fixed values, and accounts for a more general description (a set of pdfs) when these parameters are bounded. In particular, for fixed basic variables, the pdf expression corresponds to the maximum entropy distribution [13] that arises from specifying the expected values $\mathrm{E}\left[f_j(x)\right]$, where the basic variables are replaced by the Lagrange multipliers (which are constant values). It can be therefore argued that Eq. (1) represents a family of *maximum entropy continuous distributions*. When the constraints are expressed in terms of statistical expectation inequality constraints, such as:

$$v_{j,\min} \le v_j = \mathrm{E}\left[f_j(x)\right] = \int f_j(x)\, p(x|\mathbf{a})\,\mathrm{d}x \le v_{j,\max}, \quad j = 2,3,\ldots,n \tag{2}$$

where $v_{j,\min}$ and $v_{j,\max}$ are the lower and upper bound on the jth statistical expectation v_j, within a class of distribution (for example, polynomial distributions, maximum entropy distribution, etc.), there are many distributions which are consistent with the statistical expectation inequality constraints. The Principle of Maximum Entropy (MAXENT) selects, among the class of maximum entropy distributions, the distribution with the largest entropy [37]. The proposed approach, instead, constructs a family of maximum entropy distributions consistent with the statistical expectation inequality constraints and selects, among this family of pdfs, the pdf which maximises (or equivalently minimises) a specified engineering metric (for example, the probability of exceeding a specified limit value, the probability of being within a certain region). This pdf is potentially different from the pdf which maximises the entropy (which can be recovered as well); therefore the proposed approach is more useful from an engineering point of view. This aspect of the approach will be illustrated by a numerical application in Section 5 of this paper.
The inequality constraints on the statistical expectation of the uncertain variable may arise by analysing a small data set or can be provided by an expert who may prefer to assign bounds rather than specifying a single value. If $f_j(x) = x$ then the inequality constraints are specified on the mean value, alternatively if $f_j(x) = x^2$ they are specified on the second moment. $f_j(x)$ can be also defined as an interval of possible values that the uncertain variable may take, i.e. $f_j(x) = [b,c]$; in this case the constraints corresponds to the probability of finding the random variable within those bounds.
The family of pdfs defined in Eq. (1) is constructed as follows:

1. The form of the pdf which maximises the entropy is computed, as for the maximum entropy approach, by using the Lagrange multipliers method.
2. The Lagrange multipliers are substituted by the basics variables $\mathbf{a}$.
3. The bounds on the statistical expectations of the uncertain variable are used to establish bounds on the basic variables.

A procedure for obtaining an approximate mapping of the basic variables domain (a-domain) starting from a bounded description of the statistical expectations (m-domain) of the uncertain variable [36,38] is summarized in the next Subsection.

2.2 Bounds Conversion

Consider the case for which two statistical expectations of the uncertain variable x lie within a rectangle, as described in Figure 1.

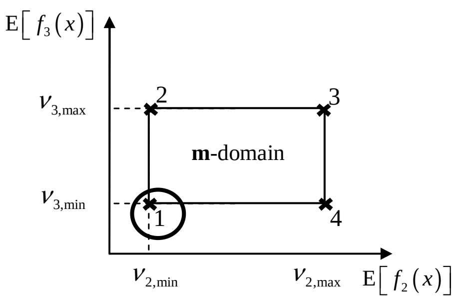

Figure 1: Moment domain (m-domain).

The first step of the approach requires the evaluation of the maximum entropy distribution, which for the present case take the form

$$p(x|\mathbf{a}) = \exp[a_1 + a_2 f_2(x) + a_3 f_3(x)]. \tag{3}$$

In principle, each point of the basic variables domain (a-domain), which is depicted in Figure 2, can be evaluated by solving a set of two non-linear equations in terms of the statistical expectations of the random variable.

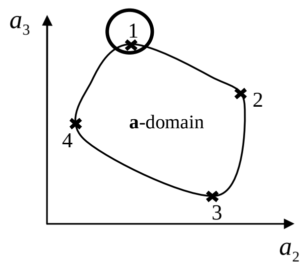

Figure 2: Basic variables domain (a-domain).

For example, point 1 of the m-domain can be mapped in the corresponding point 1 of the a-domain by solving:

$$\begin{cases} \int f_2(x)\exp\left[a_1 + a_2 f_2(x) + a_3 f_3(x)\right]\mathrm{d}\,x = v_{2,\min} \\ \int f_3(x)\exp\left[a_1 + a_2 f_2(x) + a_3 f_3(x)\right]\mathrm{d}\,x = v_{3,\min} \end{cases} \tag{4}$$

where a_2 and a_3 are the unknown coefficients, and a_1 is chosen to satisfy the normalisation condition.

In practice, considering enough points along the edges of the m-domain would allow a good approximation of the shape of the a-domain to be obtained, reducing the number of sets of equations to be solved. The problem is that, even for a simple problem (like the 2D case depicted in Figure 1), the solution of each set of non-linear equations can be time consuming and convergence problems may occur.

An approximate mapping of the a-domain can be obtained by [36,38]:

I. Evaluating the mid-points of the surfaces of the hypercube defining the m-domain ($v_j^* = \mathrm{E}\left[f_j^*(x)\right] = \left(v_{j,\max} - v_{j,\min}\right)/2$).
II. Estimate the corresponding point a^* solving a set of non-linear equations for the mid-point of the m-domain.
III. Each point of the a-domain is then calculated by using an approximate expression of the sth moment:

$$\begin{aligned} v_s = v_s^* &+ \sum_{j=2}^{n} c_j^{s*}\left(a_j - a_j^*\right) \\ &+ \frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{n} c_{jk}^{s*}\left(a_j - a_j^*\right)\left(a_k - a_k^*\right), \end{aligned} \tag{5}$$

where:

$$c_j^{s*} = \mathrm{E}\left\{\left(f_s(x) - \mathrm{E}\left[f_s^*(x)\right]\right)\left(f_j(x) - \mathrm{E}\left[f_j^*(x)\right]\right)\right\}, \tag{6}$$

$$c_{jk}^{s*} = \mathrm{E}\left\{\begin{array}{l}\left(f_s(x) - \mathrm{E}\left[f_s^*(x)\right]\right)\left(f_j(x) - \mathrm{E}\left[f_j^*(x)\right]\right) \\ \times\left(f_k(x) - \mathrm{E}\left[f_k^*(x)\right]\right)\end{array}\right\}. \tag{7}$$

This approach is expected to yield less accurate results when the variation of the sth moment value with respect to the mid-point moment domain value becomes large.

3 The Hybrid FE/SEA Method Combined with the Laplace Asymptotic Method

In this Section the hybrid FE/SEA approach and its combination with the Laplace asymptotic method are briefly reviewed.

3.1 Basic Concepts

The hybrid Finite Element/Statistical Energy Analysis (FE/SEA) method [7,34] is a vibro-acoustic analysis technique which combines the strength of a well established low-frequency deterministic technique, the Finite Element method (FE) [32], with a high-frequency probabilistic method, the Statistical Energy Analysis method (SEA) [6,33], by means of the diffuse field reciprocity relation [39,40]. With this approach, within the frequency range of interest of the problem on hand, a complex system is considered as an assembly of (i) components with very few local modes, collectively called the "master" system and modelled by using FE; and (ii) components with many local modes, called "subsystems", which are modelled with SEA, and it is assumed that all the SEA subsystems are coupled exclusively through the master system. For example, a generic class of engineering systems characterised by thin panels coupled through stiff structural components is often encountered in aerospace structures, where a frame is coupled with a skin panel, or in automotive structures, where the frame of the car is coupled to the roof panel and window panel. Within the hybrid FE/SEA modelling strategy, the panels would be modelled as SEA subsystems, and the stiff components would be modelled using FE. The response of the master system is described by a set of nodal degrees of freedom $\mathbf{q}$, and the response of the SEA subsystems is described by a set of vibrational energies $\mathbf{E}$ (defined as twice the time-averaged kinetic energy).

The properties of the hybrid FE/SEA model components (such as density, Young's modulus, geometry, etc.) are represented by two groups of parameters to distinguish different models of uncertainty [8]: the master system properties are represented by a set of parameters $\mathbf{b}$, while the properties of the SEA subsystem are represented by a set of parameters $\mathbf{s}$. The effect of the uncertain parameters $\mathbf{s}$ is accounted for via a non-parametric statistical approach based on the fact that at high frequency the statistics of the natural frequencies and mode shapes of the subsystems can approach certain universal distributions, regardless of the detailed nature of the underlying uncertainty [7,8,40]. The effect of the uncertain parameters $\mathbf{b}$ is accounted for by a probabilistic parametric uncertainty model [8]. The system is therefore varying over two ensembles: a non-parametric ensemble (a collection of systems with random subsystem properties) and a parametric ensemble (a collection of systems with random master system properties).

For fixed master system properties, the hybrid FE/SEA method enables the calculation of the conditional non-parametric ensemble average $\mu_j(\mathbf{b})$ and ensemble variance $\sigma_j^2(\mathbf{b})$ of a response variable w (which can be the vibrational energy of the SEA subsystem j, or the cross spectrum of the finite element degrees of freedom) [7,34]. The ensemble is non-parametric in the sense that the details of the parameters $\mathbf{s}$ are never considered in the model, but rather the Gaussian Orthogonal Ensemble (GOE) is used to described the statistics of the subsystem natural frequencies and mode shapes [7,40]. This approach obviates the need for any detailed knowledge of the variability or uncertainty of the parameters $\mathbf{s}$ and does not require Monte Carlo Simulations to be performed to propagate the uncertainty. The equations necessary for the evaluation of $\mu_j(\mathbf{b})$ are reviewed in the following Subsection.

3.2 The Hybrid FE/SEA Equations for Fixed FE Properties

The hybrid FE/SEA equations for evaluating the ensemble average response ($\mu_j(\mathbf{b})$) at the excitation frequency ω are [34]:

a) Subsystem energy balance equations

$$\omega(\eta_j + \eta_{d,j})E_j + \sum_k \omega\eta_{jk} n_j (E_j / n_j - E_k / n_k) = P_{in,j}^{ext} + P_{in,j}, \quad (8)$$

where η_j is the damping loss factor of the subsystem j, $\eta_{d,j}$ is an additional loss factor on the subsystem j due to the energy dissipated in the FE components, η_{jk} is the coupling loss factor between subsystem j and subsystem k, n_j is the modal density of subsystem j (which is defined as the average number the average number of natural frequencies within a unit frequency band), E_j is the ensemble average vibrational energy of subsystem j, $P_{in,j}^{ext}$ is the external power input to the subsystem arising from the loads acting on the master system and $P_{in,j}$ is the power input arising from external loads directly applied to the subsystem j.

Eq. (8) states that the power dissipated through damping ($\omega(\eta_j + \eta_{d,j})E_j$) plus the net power transmitted to other subsystems ($\sum_k \omega\eta_{jk} n_j (E_j / n_j - E_k / n_k)$) is balanced by the power input to the subsystem ($P_{in,j}^{ext} + P_{in,j}$), and it is based on the assumption that the power transmitted is proportional to the difference of the average modal energies (defined as E_j / n_j) of the coupled subsystems. Eq. (8) has the same form as the standard SEA equations [33], but also contains two additional terms relating to: (i) the contribution of the master system to the power input $P_{in,j}^{ext}$, and (ii) the power dissipated in the master system, $\omega\eta_{d,j}E_j$. These two terms can be expressed in terms of: (i) the total dynamic stiffness matrix $\mathbf{D}_{tot} = \sum_k \mathbf{D}_{dir}^{(k)} + \mathbf{D}_d$, where $\mathbf{D}_d$ is the dynamic stiffness matrix associated with the FE model ($\mathbf{D}_d = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$, where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the FE component mass, damping and stiffness matrices), and $\mathbf{D}_{dir}^{(k)}$ is the so-called direct field dynamic stiffness matrix for subsystem k which can be computed using various techniques [34]; (ii) the cross-spectral matrix of the loading applied directly to the master system $\mathbf{S}_{ff} = \left[\mathbf{f}\mathbf{f}^{*\mathrm{T}}\right]$, so that

$$\omega\eta_{d,j} = \left(\frac{2\,\alpha_k}{\pi n_j}\right) \sum_{rs} \mathrm{Im}\left\{D_{d,rs}\right\} \left(\mathbf{D}_{tot}^{-1}\,\mathrm{Im}\left\{\mathbf{D}_{dir}^{(j)}\right\}\mathbf{D}_{tot}^{-1*\mathrm{T}}\right)_{rs}, \quad (9)$$

$$P_{in,j}^{ext} = (\omega/2) \sum_{rs} \mathrm{Im}\left\{D_{dir,rs}^{(j)}\right\} \left(\mathbf{D}_{tot}^{-1}\mathbf{S}_{ff}\mathbf{D}_{tot}^{-1*\mathrm{T}}\right)_{rs}, \quad (10)$$

where the superscript $*$ indicates the complex conjugate, the superscript T denotes the transpose, Im represents the imaginary part of the matrix, and α_k is a factor which takes into account the fact that the subsystem wave field may not be perfectly diffuse [7]. Generally α_k is equal to 1 when the subsystem wave field is diffuse, and close to 2 when the subsystem is excited predominantly by motion of the master system [7].

Three of the remaining terms in Eq. (8), specifically η_j, n_j, and $P_{in,j}$, are evaluated by using standard SEA procedures [33], while the coupling loss factors are expressed analytically as a function of the total dynamic stiffness matrix in the form [34]

$$\omega\eta_{jk}n_j = \left(\frac{2\ \alpha_k}{\pi}\right)\sum_{rs}\mathrm{Im}\left\{D_{dir,rs}^{(j)}\right\}\left(\mathbf{D}_{tot}^{-1}\,\mathrm{Im}\left\{\mathbf{D}_{dir}^{(k)}\right\}\mathbf{D}_{tot}^{-1*\mathrm{T}}\right)_{rs}. \quad (11)$$

Writing Eq. (8) for each subsystem leads to a set of equations that can be solved to yield the ensemble average vibrational energy E_j of each subsystem. This set of E_j is then used to calculate the average response of the master system.

b) Master system response equation

$$\mathbf{S}_{qq} = \mathbf{D}_{tot}^{-1}\left[\mathbf{S}_{ff} + \sum_k\left(\frac{4\alpha_k E_k}{\omega\pi n_k}\right)\mathrm{Im}\left\{\mathbf{D}_{dir}^{(k)}\right\}\right]\mathbf{D}_{tot}^{-1*\mathrm{T}}, \quad (12)$$

here $\mathbf{S}_{qq}$ is the cross-spectrum of the response of the master system (averaged over the non-parametric ensemble), and the two terms on the right-hand side correspond to the forcing arising from external excitation (expressed in terms of the cross spectrum of the forces, $\mathbf{S}_{ff}$) and the forcing arising from the subsystems, as yielded by the diffuse field reciprocity relation [39,40].

By using the hybrid FE/SEA variance theory [7] it is also possible to estimate the covariance of the subsystem energies ($\mathrm{Cov}\left[\bar{E}_j, \bar{E}_k\right]$, where $\bar{E}_j = E_j / n_j$) and the variance of the cross-spectral matrix of the response of the master system ($\mathrm{Var}\left[S_{qq}\right]$) over the non-parametric ensemble, which are indicated in what follows as $\sigma_j^2(\mathbf{b})$. These equations are required in the following developments of the theory for estimating the probability density of the general response variable, but for brevity they will not be included in this paper. The reader is referred to the paper by Langley and Cotoni [7] where their full derivations can also be found.

3.3 Hybrid FE/SEA + Laplace

The hybrid FE/SEA method has been recently combined with the Laplace's method [35] (a technique used to approximate integrals expressed in the Laplace form [41]) in order to establish the failure probability of a complex built-up system with input parameters described by a combination of parametric and non-parametric probabilistic uncertainty models.

The failure probability is defined as the probability that a deterministic limit value w_0 is reached and/or exceeded by the general response variable $w = w(\mathbf{b},\mathbf{s})$ (which can be the vibrational energy of subsystem j, or the cross spectrum response of the master system). This condition can be expressed as:

$$P_f = \mathrm{P}\left[w \geq w_0\right] = \int_{w_0}^{\infty} p(w)\,\mathrm{d}\,w. \quad (13)$$

The application of the hybrid FE/SEA method for fixed $\mathbf{b}$ yields the conditional non-parametric ensemble mean and variance of the response ($\mu_j(\mathbf{b})$ and $\sigma_j^2(\mathbf{b})$, respectively), which can then be used to evaluate the probability density function of the general response variable conditional on $\mathbf{b}$, $p(w|\mathbf{b})$; for example, the pdf of the non-parametric ensemble vibrational energy is usually log-normal, and therefore the mean and variance yield the complete pdf [6,8,42]. Eq. (13) can be conveniently rewritten in terms of $p(w|\mathbf{b})$:

$$P_f = \int_{w_0}^{\infty}\int_{\mathbf{b}} p(w|\mathbf{b})\,p(\mathbf{b})\,\mathrm{d}\,\mathbf{b}\,\mathrm{d}\,w. \quad (14)$$

The failure probability conditional on $\mathbf{b}$ can be now defined as:

$$P_f(\mathbf{b}) = \int_{u_0}^{\infty} p(w|\mathbf{b})\,\mathrm{d}\,w; \quad (15)$$

and therefore Eq. (13) can be written as an unbounded integral:

$$P_f = \int_{\mathbf{b}} P_f(\mathbf{b})\,p(\mathbf{b})\,\mathrm{d}\,\mathbf{b}. \quad (16)$$

The integral in Eq. (16) can be evaluated numerically by considering a grid of integration points (direct integration), although this approach is unpractical when a large number of uncertain input parameters is considered [10]. Alternatively, an approximate evaluation of this integral can be obtained by applying the Laplace's method to the integral expressed in the form $\int_{\mathbf{b}}\exp\left[\ln\left[P_f(\mathbf{b})\,p(\mathbf{b})\right]\right]\mathrm{d}\,\mathbf{b}$. In particular, the failure probability can be approximated as [35]:

$$P_f \approx \sum_{j=1}^{\psi} P_f\left(\mathbf{b}_j^*\right)p\left(\mathbf{b}_j^*\right)(2\pi)^{d/2}\det\left[\mathbf{H}\left(\mathbf{b}_j^*\right)\right]^{-1/2}, \quad (17)$$

where ψ stands for the number of local maxima of $\ln\left[P_f(\mathbf{b})\,p(\mathbf{b})\right]$ at locations $\mathbf{b}_j^*$, d is the dimension of the set of random variables $\mathbf{b}$ involved in the problem, $\det[\ ]$ is the matrix determinant operator and $\mathbf{H}\left(\mathbf{b}_j^*\right)$ is the Hessian matrix whose elements are given by

$$H_{ij}(\mathbf{b}) = -\frac{\partial^2}{\partial b_i \partial b_j}\left[\ln\left(P_f(\mathbf{b})\,p(\mathbf{b})\right)\right]. \quad (18)$$

This approximation (Eq. (17)) corresponds to replacing the integrand function with an n-dimensional Gaussian distribution with mean equal to $\mathbf{b}_j^*$ and covariance matrix equal to the inverse of $\mathbf{H}\left(\mathbf{b}_j^*\right)$. Conditions for the accuracy of Eq. (17) are discussed in references [41,43].

4 Bounds on the Failure Probability

4.1 Hybrid FE/SEA + Laplace Using Imprecise Probabilities

The hybrid FE/SEA + Laplace approach can be generalised considering the case in which the uncertain input parameters $\mathbf{b}$ of the FE components can be subdivided into two groups: (i) a set of parameters $\hat{\mathbf{b}}$ described by a specified probability density function $p\left(\hat{\mathbf{b}}\right)$; and (ii) a set of parameters $\mathbf{b}_{imp}$ imprecisely known described in terms of bounded statistical expectations (derived from small data set or specified by an analyst). The second set of parameters $\mathbf{b}_{imp}$ can be modelled by using the imprecise probability uncertainty model presented in Section 2. With this approach, the joint pdf of the random variables $p\left(\mathbf{b}_{imp}\middle|\mathbf{a}\right)$ is expressed in the form of a maximum entropy distribution (Eq. (1)), and the bounds on the statistical expectations are converted into bounds on the so-called basic variables $\mathbf{a}$ (as described in Section 2). If these basic variables are taken to have fixed values $\mathbf{a}$, then a single pdf $p\left(\mathbf{b}_{imp}\middle|\mathbf{a}\right)$ is identified.
According to Eq. (17), the failure probability conditional on the basic variables is then given by

$$\begin{aligned} P_f\left(\mathbf{a}\right) &= \mathrm{P}\left[w\left(\hat{\mathbf{b}}, \mathbf{b}_{imp}\middle|\mathbf{a},\mathbf{s}\right) \geq w_0 \right] \\ &= \int_{\mathbf{b}} P_f\left(\hat{\mathbf{b}}, \mathbf{b}_{imp}\middle|\mathbf{a}\right) p\left(\hat{\mathbf{b}}\right) p\left(\mathbf{b}_{imp}\middle|\mathbf{a}\right) \mathrm{d}\,\mathbf{b}. \end{aligned} \tag{19}$$

where $P_f\left(\hat{\mathbf{b}}, \mathbf{b}_{imp}\middle|\mathbf{a}\right)$ is the failure probability conditional on $\left(\hat{\mathbf{b}}, \mathbf{b}_{imp}\middle|\mathbf{a}\right)$.

The hybrid FE/SEA + Laplace approach [35] can be then employed to estimate the failure probability as:

$$\begin{aligned} P_f\left(\mathbf{a}\right) \approx & \sum_{j=1}^{\psi} P_f\left(\hat{\mathbf{b}}_j^*, \mathbf{b}_{imp,j}^*\middle|\mathbf{a}\right) p\left(\hat{\mathbf{b}}_j^*\right) p\left(\mathbf{b}_{imp,j}^*\middle|\mathbf{a}\right) \\ & \times (2\pi)^{d/2} \det\left[\mathbf{H}\left(\hat{\mathbf{b}}_j^*, \mathbf{b}_{imp,j}^*\middle|\mathbf{a}\right) \right]^{-1/2} \end{aligned} \tag{20}$$

The evaluation of the failure probability requires:

I. Evaluation of $p\left(w\middle|\left(\hat{\mathbf{b}}_j, \mathbf{b}_{imp,j}\middle|\mathbf{a}\right)\right)$ by using the results yielded by the hybrid FE/SEA method.
II. Calculation of $P_f\left(\hat{\mathbf{b}}_j, \mathbf{b}_{imp,j}\middle|\mathbf{a}\right)$ by using Eq. (15).
III. Evaluation of $\left(\hat{\mathbf{b}}_j^*, \mathbf{b}_{imp,j}^*\middle|\mathbf{a}\right)$ by applying a standard unconstrained minimization algorithm to $-\ln\left[P_f\left(\hat{\mathbf{b}}_j^*, \mathbf{b}_{imp,j}^*\middle|\mathbf{a}\right) p\left(\hat{\mathbf{b}}_j^*\right) p\left(\mathbf{b}_{imp,j}^*\middle|\mathbf{a}\right) \right]$.
IV. Evaluation of the Hessian matrix.

If the basic variables are allowed to vary, a family of response pdfs is obtained and the bounds on the failure probability can be established as

$$\min_{\mathrm{S}}\left(P_f\left(\mathbf{a}\right)\right) \leq P_f \leq \max_{\mathrm{S}}\left(P_f\left(\mathbf{a}\right)\right). \tag{21}$$

These bounds give an indication of the sensitivity of the system reliability with respect to the uncertainty on the pdf of the input parameters. If the bounds are wide, the uncertainty in the input parameter description is significantly affecting the system reliability. On the other hand, if the bounds are narrow then the system reliability is little affected by the uncertainty in the pdf of the uncertain parameters.

4.2 Steps for Implementing the Proposed Approach

The reliability analysis can be summarised as follows:

I. The system is subdivided into: (i) FE components with uncertain properties $\mathbf{b}$; and (ii) SEA components with uncertain properties $\mathbf{s}$.
II. The effect of the uncertain parameters $\mathbf{s}$ of the SEA components is accounted for by using non-parametric statistical methods.
III. The uncertain parameters of the FE components $\mathbf{b}$ are partitioned into two sets of parameters: (i) $\hat{\mathbf{b}}$ modelled by using a specified pdf $p\left(\hat{\mathbf{b}}\right)$; and (ii) $\mathbf{b}_{imp}$ modelled via the imprecise probability model $p\left(\mathbf{b}_{imp}\middle|\mathbf{a}\right)$ where $\mathbf{a}$ are the basic variables which define the family of pdfs (Eq. (1)).
IV. The admissible region of the basic variable a-domain) associated to the random variables $\mathbf{b}_{imp}$ (obtained as described in Section 2 from the knowledge of the bounds on statistical expectations) is overlaid with a grid of points. This grid is chosen in order to capture enough sampled points within and along the a-domain.
V. For each sampled point of this grid, the corresponding a_1 is calculated via normalization. The set of basic variables associated to each point of the domain identifies a single $p\left(\mathbf{b}_{imp}\middle|\mathbf{a}\right)$.
VI. For fixed basic variable $\mathbf{a}$, $P_f\left(\mathbf{a}\right)$ is calculated using Eq. (20).
VII. The bounds on the failure probability are then calculated by using Eq. (21).

5 Numerical Application

The example system is composed by two simply supported plates coupled via a spring/mass system in order to represent with the simplest possible dynamic model a generic class of systems in which thin panels are coupled to stiff structural components (such as the frame of a car coupled to the roof and the window panels). The coupling is realised using three springs attached in the interior of each plate (point connections) linked to the second mass of the spring/mass system (Figure 3). The system is excited with a unit force applied to the first mass of the spring/mass system. The two plates are made of aluminium (Young's modulus $Y = 71\times10^9\ N/m^2$, density $2700\ Kg/m^3$ and Poisson's ratio $\nu = 0.3$) and their properties are summarised in Table 1.

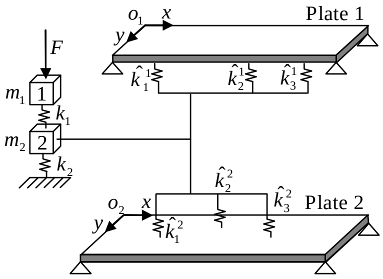

Figure 3: Built-up plate system under investigation.

Elements	Thickness (mm)	Size $L_x \times L_y$ ($m \times m$)	Loss factor η (%)	Modal density n (modes/Hz)
Plate 1	1.25	1.4×1.2	2	0.4286
Plate 2	1.25	1.4×1.3	2	0.4643

Table 1: Properties of the plates.

The spring connections in the interior of the first plate have stiffness $\hat{k}_u^1 = 2\times10^6$ N/m, $(u = 1,2,3)$ and attachment points $(0.3, 0.8), (0.6, 0.4)$, and $(0.8, 0.6)$ measured in metres along the x and y directions and relative to point the o_1. The second plate is connected via springs of stiffness $\hat{k}_l^2 = 2\times10^4$ N/m, $(l = 1,2,3)$ attached at points $(0.4, 0.4), (0.5, 0.9)$, and $(0.9, 0.7)$ measured in metres along the x and y directions and relative to the point o_2.

The hybrid FE/SEA model of the system comprises two SEA subsystems (the plates), which are highly random, and a mass/spring system (FE component) with two uncertain parameters, namely k_1 and k_2. k_1 is described by a lognormal pdf with mean value 6×10^6 N/m and variance $10^{11}\,(\text{N/m})^2$. k_2 is imprecisely known and it is specified in terms of bounds on statistical expectations as summarised in Table 2 and depicted in Figure 4.

1	2	3	4
$(18\times10^5, 14.27)$	$(22\times10^5, 14.52)$	$(22\times10^5, 14.50)$	$(18\times10^5, 14.24)$

Table 2: Coordinates of the vertices of the m-domain.

The system is forced by a unit force applied to the first mass of the mass/spring system (as shown in Figure 3). The design target is the energy level of plate 1 at 145 Hz, and a limiting value of $E_0 = 0.02\times10^{-4}\ J$ is considered.

The initial step of the analysis consists of evaluating the probability density function of the uncertain parameter k_2. This is achieved by using the procedure described in Subsection 2.1. The pdf of k_2 has the form

$$p(k_2\,|\,\mathbf{a}) = \exp[a_1 - a_2 x - a_3 \ln(x)], \tag{22}$$

where a_1 is obtained by using the normalization condition as:

$$a_1 = -\ln\left(a_2^{(a_3-1)}\Gamma(1-a_3)\right), \tag{23}$$

where $\Gamma(\bullet)$ is the gamma function.

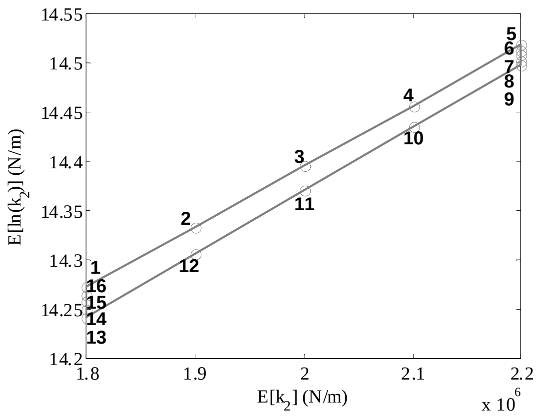

Figure 4: Moment domain for k_2

The a-domain is then calculated by using the strategy summarized in Subsection 2.2. In particular, the quadratic approximation of statistical expectations (Eq. (5)) was employed and 16 points along the m-domain (as shown in Figure 4) were mapped into the a-domain. The resulting approximate domain is shown in Figure 5.

Each point of the a-domain defines a single pdf. Some of the pdfs corresponding to the a-domain are shown in Figure 6.

The second step of the analysis consists of approximating the bounds on the failure probability as described in Subsection 4.2.

The a-domain was overlaid with a grid of 50×50 equally-spaced points. The 16 points along the domain and 414 points internal to the domain were considered (for a total of 430 pdfs). For each grid point (a_2, a_3) the procedure illustrated in Subsection 4.1 was applied. In particular, for fixed $(\hat{\mathbf{b}}_j, \mathbf{b}_{imp,j}|\mathbf{a})$ the hybrid FE/SEA method was applied to estimate the mean and variance of the response. These were used, under the assumption of a lognormal distribution of the vibrational energy of plate 1, to evaluate $p\left(w|(\hat{\mathbf{b}}_j, \mathbf{b}_{imp,j}|\mathbf{a})\right)$. $P_f(\hat{\mathbf{b}}_j, \mathbf{b}_{imp,j}|\mathbf{a})$ was then calculated by using Eq. (15). The minimum point(s) of $-\ln\left[P_f(\hat{\mathbf{b}}_j^*, \mathbf{b}_{imp,j}^*|\mathbf{a})\, p(\hat{\mathbf{b}}_j^*)\, p(\mathbf{b}_{imp,j}^*|\mathbf{a})\right]$ was calculated by using the Matlab function fminunc. The Hessian matrix was approximated by using third order Lagrange polynomials. Finally, the failure probability conditional on the basic variables was computed by using Eq. (20).

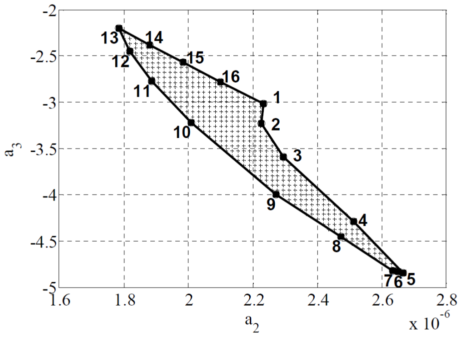

Figure 5: Approximate a-domain.

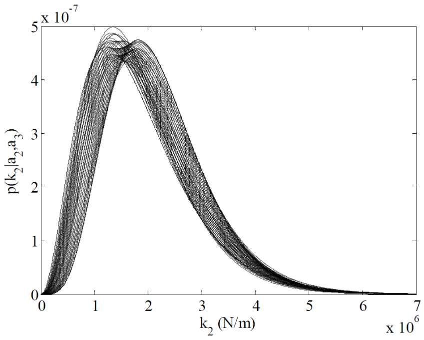

Figure 6: Pdfs generated from the a-domain.

The results obtained for each grid point are shown in Figure 7.

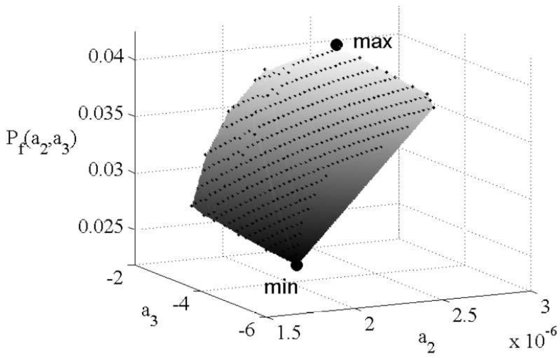

Figure 7: Failure probability as a function of the basic variables. The lower and upper bounds of the failure probability are labeled as "min" and "max".

The bounds on the failure probability are (by using Eq. (21)): $0.02192 \le P_f \le 0.04245$ (respectively, at point 1 and 9 of the a-domain), meaning that the uncertainty in the input parameters significantly affects the failure probability estimates. The computational time required by the proposed approach was of about 3 minutes.

The failure probability obtained for the MAXENT distribution (corresponding to the point 10 of the a-domain in Figure 7) is 0.03976. The MAXENT distribution would therefore underestimate the maximum failure probability.

The results obtained with the proposed approach were validated against direct numerical integration of Eq. (19), which took about 6 hours, showing differences less than 1%. Full FE Monte Carlo simulations for the present system considering a single point (and therefore a single pdf) of the a-domain requires about 45 hours. Full FE Monte Carlo simulations are therefore unfeasible even for this example system. It can be concluded that the proposed approach provides a very efficient tool for the reliability analysis of system with uncertain properties.

6 Summary and Conclusions

An imprecise probability model based on expressing the pdf of a random variable in the form of a maximum entropy distribution with bounded parameters was used to describe the parametric uncertainty of the FE components of a hybrid FE/SEA model. The hybrid FE/SEA + Laplace method, which fully accounts for both parametric (FE components) and non-parametric (SEA components) uncertainties, was applied to establish bounds on the failure probability. These bounds give an indication of the sensitivity of the system reliability to the uncertain input parameters and allow establishing the highest failure probability expected.

This approach provides a very useful tool for evaluating the reliability of complex engineering systems given that:

- The partition of the system in SEA and FE components leads to a large reduction of the number of degrees of freedom employed in the model (potentially thousand of finite elements nodes are substituted with a single degree of freedom SEA subsytem) and a large gain in numerical efficiency.
- The SEA subsystem ensemble is dealt with analytically (without using MCS) leading to a further reduction in computational costs.
- The uncertainty in FE components is dealt with using the Laplace asymptotic method instead of MCS.
- The bounds on the failure probability can be efficiently established when the imprecise probability model is employed.

The method has been illustrated by application to built-up plate systems, showing a large reduction of the computational cost when compared to a direct integration procedure and to Full FE Monte Carlo simulations.

Acknowledgements

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