# Inclusion/exclusion principle for belief functions

**Felipe Aguirre**[1]
felipe.aguirre@utc.fr

**Christelle Jacob**[2]
jacob@isae.fr

**Sébastien Destercke**[1]
sebastien.destercke@utc.fr

**Didier Dubois**[3]
dubois@irit.fr

**Mohamed Sallak**[1]
mohamed.sallak@utc.fr

## Abstract

The inclusion-exclusion principle is a well-known property of set cardinality and probability measures, that is instrumental to solve some problems such as the evaluation of systems reliability or of uncertainty over Boolean formulas. However, when using sets and probabilities conjointly, this principle no longer holds in general. It is therefore useful to know in which cases it is still valid. This paper investigates this question when uncertainty is modelled by belief functions. After exhibiting necessary and sufficient conditions for the principle to hold, we illustrate its use on some applications, i.e. reliability analysis and uncertainty over Boolean formulas. [1]

## 1 Introduction

Probability theory is the most well-known approach to model uncertainty. However, even when the existence of a single probability is assumed, it often happens that the distribution is partially known, in which case one is forced to use a selection principle (e.g., maximum entropy [13]) to work within probability theory. This is particularly the case in the presence of severe uncertainty (few samples, imprecise or unreliable data, ...) or when subjective beliefs are elicited. Many authors claim that in situations involving imprecision or incompleteness, uncertainty cannot be modelled faithfully by a single probability, and they have proposed frameworks to properly model such uncertainty: possibility theory [11], belief functions [16], imprecise probabilities [17], info-gap theory [3], ...

A known practical drawback of belief functions and of other imprecise probabilistic theories is that their manipulation can be computationally more demanding than probabilities. Indeed, the fact that belief functions are more general than classical probabilities prevents the use of

some properties that hold for the latter but not for the former. This is the case, for instance, of the well known and useful inclusion-exclusion principle (also known as the sieve formula or Sylvester-Poincaré equality).

Given a space $\mathscr{X}$, a probability measure $P$ over this space and a collection $\mathscr{A}_N = \{A_1,\ldots,A_N | A_i \subseteq \mathscr{X}\}$ of measurable subsets of $\mathscr{X}$, the inclusion-exclusion principle states that

$$P(\cup_{i=1}^n A_i) = \sum_{\mathscr{I} \subseteq \mathscr{A}_n} (-1)^{|\mathscr{I}|+1} P(\cap_{A \in \mathscr{I}} A) \qquad (1)$$

where $|\mathscr{I}|$ is the cardinality of $\mathscr{I}$. This equality allows to easily compute the probability of $\cup_{i=1}^n A_i$. This principle is used in numerous problems, including the evaluation of the reliability of complex systems when using minimal paths.

In this paper, we investigate in Section 2 necessary and sufficient conditions under which a similar equality holds for belief functions. Section 3 then studies how the results apply to the practically interesting case where events $A_i$ and focal sets are Cartesian products. Section 4 then shows that such conditions are met for specific events of monotone functions, and applies this result to the reliability analysis of multi-state systems. Finally, Section 5 computes the belief and plausibility of Boolean formulas expressed in normal forms.

## 2 General Additivity Conditions for Belief Functions

After introducing notations, Section 2.2 provides general conditions for families of subsets for which the inclusion-exclusion principle holds for belief functions. We then interest ourselves to the specific case where focal sets of belief functions are Cartesian products of subsets.

### 2.1 Setting

A mass distribution [16] defined on a (finite) space $\mathscr{X}$ is a mapping $m : 2^{\mathscr{X}} \to [0,1]$ from the power set of $\mathscr{X}$ to the

unit interval such that $m(\emptyset) = 0$ and $\sum_{E \subseteq \mathscr{X}} m(E) = 1$. A set $E$ that receives a strictly positive mass is called *focal set*, and the set of focal sets of $m$ is denoted by $\mathscr{F}_m$. From the mapping $m$ are usually defined two set-functions, the plausibility and the belief functions, respectively defined for any $A \subseteq \mathscr{X}$ as

$$Pl(A) = \sum_{E \cap A \neq \emptyset} m(E), \qquad (2)$$

$$Bel(A) = \sum_{E \subseteq A} m(E) = 1 - Pl(A^c). \qquad (3)$$

They are such that $Bel(A) \leq Pl(A)$. The plausibility function measures how much event $A$ is possible, while the belief function measures how much event $A$ is certain. In the theory of evidence [16], belief and plausibility functions are interpreted as confidence degrees not necessarily related to probabilities. However, the mass distribution $m$ can also be interpreted as the random set corresponding to an imprecisely observed random variable [8], in which case $Bel, Pl$ can be interpreted as probability bounds inducing a convex set $\mathscr{P}(Bel)$ such that

$$\mathscr{P}(Bel) = \{P | \forall A, Bel(A) \leq P(A) \leq Pl(A)\}$$

is the set of all probabilities bounded by $Bel$ and $Pl$. Note that, since $Bel$ and $Pl$ are dual ($Bel(A) = 1 - Pl(A^c)$), we can concentrate on one of them. A distribution $m$ can be seen as a probability distribution over sets, and in this sense it captures both probabilistic and set-based modelling: any probability $p$ can be modelled by a mass $m$ such that $m(\{x\}) = p(x)$ and any set $E$ can be modelled by the mass $m(E) = 1$.

Consider now a collection of events $\mathscr{A}_n = \{A_1, \ldots, A_n | A_i \subseteq \mathscr{X}\}$ of subsets of $\mathscr{X}$ and a mass distribution $m$ from which can be computed a belief function $Bel$. Usually, we have the inequality [16]

$$Bel(\cup_{i=1}^n A_i) \geq \sum_{\mathscr{I} \subseteq \mathscr{A}_n} (-1)^{|\mathscr{I}|+1} Bel(\cap_{A \in \mathscr{I}} A) \qquad (4)$$

that is to be compared to Eq. (1). Belief functions are said to be *n-monotonic* for any $n > 0$. Note that we can assume without loss of generality that for any $i, j$, $A_i \nsubseteq A_j$ (otherwise $A_i$ can be suppressed from Equation 4), that is there is no inclusion between the sets of $\mathscr{A}_n$. If Equation 4 becomes an equality, we will say that the belief is *additive* for collection $\mathscr{A}_n$, or $\mathscr{A}_n$-additive for short.

## 2.2 General necessary and sufficient conditions

In the case of two events $A_1$ and $A_2$, none of which is included in the other one, the basic condition for the inclusion-exclusion law to hold is that focal sets included in $A_1 \cup A_2$ should only lie (be included) in $A_1$ or $A_2$. Indeed, otherwise, if $\exists E \nsubseteq A_1$ and $E \nsubseteq A_2$ with $m(E) > 0$,

then

$$Bel(A_1 \cup A_2) \geq m(E) + Bel(A_1) + Bel(A_2) - Bel(A_1 \cap A_2)$$
$$> Bel(A_1) + Bel(A_2) - Bel(A_1 \cap A_2).$$

So, one must check that $\mathscr{F}_m$ satisfies:

$$\mathscr{F}_m \cap 2^{A_1 \cup A_2} = \mathscr{F}_m \cap \left(2^{A_1} \cup 2^{A_2}\right)$$

where $2^C$ denote the set of subsets of $C$. So, one must check that $\forall E \in \mathscr{F}_m$ such that $E \subseteq A_1 \cup A_2$, either $E \subseteq A_1$ or $E \subseteq A_2$, or equivalently

**Lemma 1.** *A belief function is additive for $\{A_1, A_2\}$ if and only if $\forall E \subseteq A_1 \cup A_2$ such that $(A_1 \setminus A_2) \cap E \neq \emptyset$ and $(A_2 \setminus A_1) \cap E \neq \emptyset$ then $m(E) = 0$.*

*Proof.* Immediate, as $E$ overlaps $A_1$ and $A_2$ without being included in one of them if and only if $(A_1 \setminus A_2) \cap E \neq \emptyset$ and $(A_2 \setminus A_1) \cap E \neq \emptyset$. $\qquad \square$

This result can be extended to larger collections of sets $\mathscr{A}_n, n > 2$ in quite a straightforward way

**Proposition 1.** $\mathscr{F}_m$ *satisfies the property $\mathscr{F}_m \cap 2^{A_1 \cup \ldots \cup A_n} = \mathscr{F}_m \cap \left(2^{A_1} \cup \ldots \cup 2^{A_n}\right)$ if and only if $\forall E \subseteq (A_1 \cup \ldots \cup A_n)$, if $E \in \mathscr{F}_m$ then $\nexists A_i, A_j$ such that $(A_i \setminus A_j) \cap E \neq \emptyset$ and $(A_j \setminus A_i) \cap E \neq \emptyset$.*

*Proof.* $\mathscr{F}_m \cap 2^{A_1 \cup \ldots \cup A_n} = \mathscr{F}_m \cap \left(2^{A_1} \cup \ldots \cup 2^{A_n}\right)$
if and only if $\nexists E \in \mathscr{F}_m \cap \left(2^{A_1 \cup \ldots \cup A_n} \setminus \left(2^{A_1} \cup \ldots \cup 2^{A_n}\right)\right)$
if and only if $\nexists E \subseteq (A_1 \cup \ldots \cup A_n), E \in \mathscr{F}_m$ such that $\forall i = 1, \ldots, n, E \nsubseteq A_i$
if and only if $\nexists i \neq j, E \in \mathscr{F}_m, E \nsubseteq A_i, E \nsubseteq A_j, E \cap A_i \neq \emptyset, E \cap A_j \neq \emptyset$
if and only if $\nexists i \neq j, E \in \mathscr{F}_m$, with $(A_i \setminus A_j) \cap E \neq \emptyset$ and $(A_j \setminus A_i) \cap E \neq \emptyset$ $\qquad \square$

So, based on Proposition 1, we have:

**Theorem 2.** *The equality*

$$Bel(\cup_{i=1}^n A_i) = \sum_{\mathscr{I} \subseteq \mathscr{A}_n} (-1)^{|\mathscr{I}|+1} Bel(\cap_{A \in \mathscr{I}} A) \qquad (5)$$

*holds if and only if $\forall E \subseteq (A_1 \cup \ldots \cup A_n)$, if $m(E) > 0$, then $\nexists A_i, A_j$ such that $(A_i \setminus A_j) \cap E \neq \emptyset$ and $(A_j \setminus A_i) \cap E \neq \emptyset$.*

Theorem 2 shows that going from $\mathscr{A}_2$-additivity for 2 given sets to $\mathscr{A}_n$-additivity is straightforward, as ensuring $\mathscr{A}_n$-additivity comes down to checking the additivity conditions for every pair of subsets in $\mathscr{A}$.

Note that by duality one also can write a form of inclusion-exclusion property for plausibility functions:

$$Pl(\cap_{i=1}^n B_i) = \sum_{\mathscr{I} \subseteq \mathscr{B}_n} (-1)^{|\mathscr{I}|+1} Pl(\cup_{B \in \mathscr{I}} B) \qquad (6)$$

for a family of sets $\mathscr{B}_n = \{\overline{A}_i : A_i \in \mathscr{A}_n\}$ where $\mathscr{A}_n$ satisfies the condition of Proposition 1.

# 3 When focal sets are Cartesian products

In this section, we investigate a practically important sub-case where focal sets and events $A_i, i = 1, \ldots, n$ are Cartesian products. That is, we assume that $\mathscr{X} = \mathscr{X}^1 \times \ldots \times \mathscr{X}^D := \mathscr{X}^{1:D}$ is the product space of *finite* spaces $\mathscr{X}^i$, $i = 1, \ldots, D$. We will call the spaces $\mathscr{X}^i$ *dimensions*. We will denote by $X_i$ the value of a variable (e.g., the state of a component, the value of a propositional variable) on $\mathscr{X}^i$.

Given $A \subseteq \mathscr{X}$, we will denote by $A^i$ the projection of $A$ on $\mathscr{X}^i$. Let us call *rectangular* a subset $A \subseteq \mathscr{X}$ that can be expressed as the Cartesian product $A = A^1 \times \ldots \times A^D$ of its projections (in general, we only have $A \subseteq A^1 \times \ldots \times A^D$ for any subset $A$). Note that a rectangular subset $A$ is completely characterized by its projections.

In the following we study the additivity property for families $\mathscr{A}_n$ containing rectangular sets only, when the focal sets of mass functions defined on $\mathscr{X}$ are also rectangular (to simplify the proofs, we will also assume that all rectangular sets are focal sets). Note that, in practice, assuming sets of $\mathscr{A}$ to be rectangular is not very restrictive, as in the finite case, any set $A \subseteq \mathscr{X}$ can be decomposed into a union of rectangular subsets.

## 3.1 Two sets, two dimensions

Let us first explore the case $n = 2$ and $D = 2$, that is $\mathscr{A}_2 = \{A_1, A_2\}$ with $A_i = A_i^1 \times A_i^2$ for $i = 1, 2$. The main idea in this case is that if $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are rectangular with disjoint projections, then $\mathscr{A}_2$-additivity holds for belief functions and this is characteristic.

**Lemma 2.** *If $A$ and $B$ are rectangular and have disjoint projections, then there is no rectangular subset of $A \cup B$ overlapping both $A$ and $B$*

*Proof.* Consider $C = C^1 \times C^2$ overlapping both $A$ and $B$. So there is $a^1 \times a^2 \in A \cap C$ and $b^1 \times b^2 \in B \cap C$. Since $C$ is rectangular, $a^1 \times b^2$ and $b^1 \times a^2 \in C$. However if $C \subseteq A \cup B$ then $a^1 \times b^2 \in A \cup B$ and either $b^2 \in A^2$ or $a^1 \in B^1$. Since $a^1 \in A^1$ and $b^2 \in B^2$ by assumption, we reach a contradiction since projections are not disjoint. $\square$

We can now study characteristic conditions for additivity for belief functions on two sets:

**Theorem 3.** *Additivity applied to $\mathscr{A}_2 = \{A_1, A_2\}$ holds for belief functions if and only if one of the following condition holds*

1. $A_1^1 \cap A_2^1 = A_1^2 \cap A_2^2 = \emptyset$

2. $A_1^1 \subseteq A_2^1$ and $A_2^2 \subseteq A_1^2$ *(or changing both inclusion directions)*
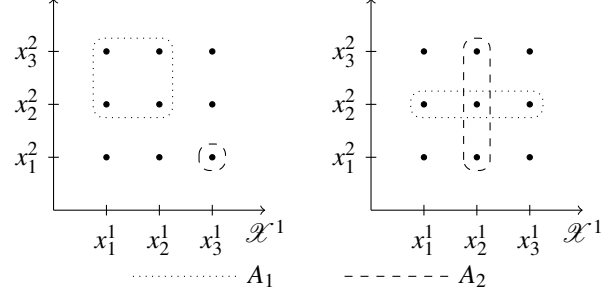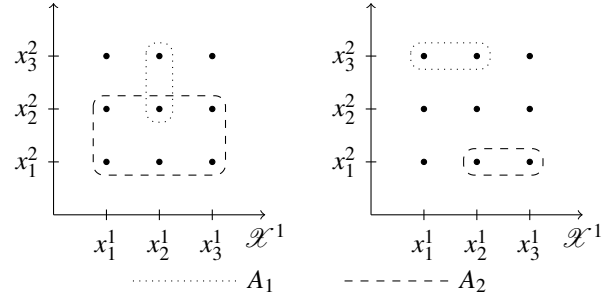


Figure 1: Situations satisfying Theorem 3



Figure 2: Situations not satisfying Theorem 3

*Proof.* First note that inclusions of Condition 2 can be considered as strict, as we have assumed $A_1, A_2$ to not be included in each other (otherwise the result is trivial).
$\Leftarrow$ 1.: If $A_1^1 \cap A_2^1 = A_1^2 \cap A_2^2 = \emptyset$, $A_1$ and $A_2$ are disjoint, as well as their projections. Then by Lemma 2 additivity holds for belief functions on any two sets.
$\Leftarrow$ 2.: $A_1^1 \subset A_2^1$ and $A_2^2 \subset A_1^2$ implies that $A_1 \setminus A_2 = A_1^1 \times (A_1^2 \setminus A_2^2)$ and $A_2 \setminus A_1 = (A_2^1 \setminus A_1^1) \times A_2^2$. As they are rectangular and have disjoint projections, Lemma 2 applies.
$\Rightarrow$ 1.: Suppose $A_1 \cap A_2 = \emptyset$ with $A_1^1 \cap A_2^1 \neq \emptyset$. Then $(A_1^1 \cap A_2^1) \times (A_1^2 \cup A_2^2)$ is rectangular, not contained in $A_1$ nor $A_2$ but contained in $A_1 \cup A_2$, so additivity does not hold.
$\Rightarrow$ 2.: Suppose $A_1^1 \subset A_2^1$ but $A_2^2 \not\subset A_1^2$. Again, $(A_1^1 \cap A_2^1) \times (A_1^2 \cup A_2^2) = A_1^1 \times (A_1^2 \cup A_2^2)$ is rectangular, neither contained in $A_1$ nor $A_2$ but contained in $A_1 \cup A_2$. $\square$

Figure 1 and 2 show various situations where conditions of Theorem 3 are satisfied and not satisfied, respectively.

## 3.2 The multidimensional case

We can now proceed to extend Theorem 3 to the case of any number $D$ of dimensions. However, this extension will not be as straightforward as going from Lemma 1 to Proposition 1, and we need first to characterize when the union of two disjoint singletons is rectangular. We will call such rectangular unions *minimal rectangles*. A singleton is a degenerated example of minimal rectangle.

**Lemma 3.** *Let $a = \{a^1\} \times \ldots \times \{a^D\}$ and $b = \{b^1\} \times \ldots \times \{b^D\}$ be two distinct singletons in $\mathscr{X}$. Then, $a \cup b$ forms a*

*minimal rectangle if and only if there is only one $i \in [1,D]$ such that $a^i \neq b^i$*

*Proof.* $\Rightarrow$: If $a^i \neq b^i$ for only one $i$, then $a \cup b = \{a^1\} \times \ldots \times \{a^i, b^i\} \times \ldots \{a^D\}$ is rectangular.

$\Leftarrow$: Consider the case where singletons differ on two components, say $a^1 \neq b^1$ and $a^2 \neq b^2$, without loss of generality. In this case,

$$a \cup b = \{\{a^1\} \times \{a^2\} \times \{a^3\} \times \ldots \times \{a^D\},$$
$$\{b^1\} \times \{b^2\} \times \{a^3\} \times \ldots \times \{a^D\}\}.$$

The projections of $a \cup b$ on dimensions 1 and 2 of $\mathscr{X}$ are $\{a^1, b^1\}$ and $\{a^2, b^2\}$ respectively, $\{a^i\}$ for $i > 2$. Hence, the Cartesian product of the projections of $a \cup b$ is the set $\{a^1, b^1\} \times \{a^2, b^2\} \times \{a^3\} \times \ldots \times \{a^D\}$. It contains elements not in $a \cup b$ (e.g. $\{a^1\} \times \times \{b^2\} \times \{a^3\} \times \ldots \times \{a^D\}$). Since $a \cup b$ is not characterised by its projections on dimensions $\mathscr{X}_i$, it is not rectangular, and this finishes the proof. $\qquad\square$

As mentioned before, any set can be decomposed into rectangular sets, and in particular any rectangular set can be decomposed into minimal rectangles. Let us now show how Theorem 3 can be extended to $D$ dimensions.

**Theorem 4.** *Additivity holds on $\mathscr{A}_2 = \{A_1, A_2\}$ for belief functions if and only if one of the following condition holds*

1. *$\exists$ distinct $p, q \in \{1, \ldots, D\}$ such that $A_1^p \cap A_2^p = A_1^q \cap A_2^q = \emptyset$*

2. *$\forall i \in \{1, \ldots, D\}$ either $A_1^i \subseteq A_2^i$ or $A_2^i \subseteq A_1^i$*

*Proof.* Again, we can consider that there are at least two distinct $p, q \in \{1, \ldots, D\}$ such that inclusions $A_1^p \subset A_2^p$ and $A_2^q \subset A_1^q$ of Condition 2 are strict, as we have assumed $A_1, A_2$ to not be included in each other (otherwise the result is trivial).

$\Leftarrow$ 1.: Any two singletons $a_1 \in A_1$ and $a_2 \in A_2$ will be such that $a_1^i \in A_1^i$ and $a_2^i \in A_2^i$ must be distinct for $i = p, q$ since $A_1^p \cap A_2^p = A_1^q \cap A_2^q = \emptyset$. Thus it will be impossible to create minimal rectangles included in $A_1 \cup A_2$, and therefore any rectangular set in it.

$\Leftarrow$ 2.: Let us denote by $P$ the set of indices $p$ such that $A_1^p \subset A_2^p$ and by $Q$ the set of indices $q$ such that $A_2^q \subset A_1^q$. Now, let us consider two singletons $a_1 \in A_1 \setminus A_2$ and $a_2 \in A_2 \setminus A_1$. Then

- $\exists p \in P$ such that $a_1^p \in A_1^p \setminus A_2^p$, otherwise $a_1$ is included in $A_1 \cap A_2$

- $\exists q \in Q$ such that $a_2^q \in A_2^q \setminus A_1^q$, otherwise $a_2$ is included in $A_1 \cap A_2$

but since $a_1^q \in A_1^q$ and $a_2^p \in A_2^p$ by definition, $a_1$ and $a_2$ must differ at least on two dimensions, hence one cannot form a minimal rectangle not in $A_1 \cap A_2$.

$\Rightarrow$ 1: Suppose $A_1 \cap A_2 = \emptyset$ with $A_1^q \cap A_2^q \neq \emptyset$ only for $q$. Then the following rectangular set contained in $A_1 \cup A_2$

$$(A_1^1 \cap A_2^1) \times \cdots \times (A_1^{q-1} \cap A_2^{q-1}) \times (A_1^q \cup A_2^q)$$
$$\times (A_1^{q+1} \cap A_2^{q+1}) \ldots \times (A_1^D \cap A_2^D) \qquad (7)$$

is neither contained in $A_1$ nor $A_2$, so additivity will not hold.

$\Rightarrow$ 2.: suppose $A_1 \cap A_2 \neq \emptyset$ and $A_1^q \nsubseteq A_2^q$, $A_1^q \nsupseteq A_2^q$ for some $q$. Again, the set (7) is rectangular, neither contained in $A_1$ nor $A_2$ but contained in $A_1 \cup A_2$. $\qquad\square$

Using Proposition 1, the extension of $\mathscr{A}_n$-additivity to $D$-dimensional sets is straightforward:

**Theorem 5.** *Additivity holds on $\mathscr{A}_N = \{A_1, \ldots, A_N\}$ for belief functions if and only if, for each pair $A_i, A_j$, one of the following condition holds*

1. *$\exists$ distinct $p, q \in \{1, \ldots, D\}$ such that $A_i^p \cap A_j^p = A_i^q \cap A_j^q = \emptyset$*

2. *$\forall \ell \in \{1, \ldots, D\}$ either $A_i^\ell \subseteq A_j^\ell$ or $A_j^\ell \subseteq A_i^\ell$*

### 3.3 On the practical importance of rectangular focal sets

While limiting ourselves to rectangular subsets in $\mathscr{A}$ is not especially restrictive, the assumption that focal sets have to be restricted to rectangular sets may seem restrictive (as it is not allowed to cut any focal set into smaller rectangular subsets without redistributing the mass). However, such mass assignments on rectangular sets are found in many practical situations:

- such masses can be obtained by defining marginal masses $m^i$ on each space $\mathscr{X}^i$, $i = 1, \ldots, D$ and then combining them under an assumption of (random set) independence [7]. In this case, the joint mass $m$ assigns to each rectangular set $E$ the mass

$$m(E) = \prod_{i=1}^{D} m^i(E^i). \qquad (8)$$

Additionally, computing belief and plausibility functions of any rectangular set $A$ becomes easier in this case, as

$$Bel(A) = \prod_{i=1}^{D} Bel^i(A^i), \quad Pl(A) = \prod_{i=1}^{D} Pl^i(A^i), \quad (9)$$

where $Bel^i, Pl^i$ are the measures induced by $m^i$;

- as all we need is to restrict masses to product events, we can also consider cases of unknown independence or of partially known dependence, as long as this knowledge can be expressed by linear constraints on the marginal masses [1];

- using more generic models than belief functions is possible [9], since the mass positivity assumption can be dropped without modifying our results.

# 4 Inclusion-exclusion for monotone functions

In this section, we show that the inclusion-exclusion principle can be applied to evaluate some events of interest for monotone functions, and we provide an illustration from Multi-State Systems (MSS) reliability.

## 4.1 Checking the conditions

Let $\phi : \mathscr{X}^{1:D} \to \mathscr{Y}$ be a $D$-placed function, where $\mathscr{X}^j = \{x_1^j, \ldots, x_{k_j}^j\}$ is a finite ordered set, for every $j = 1, \ldots, D$. We note $\leq_j$ the order relation on $\mathscr{X}_j$ and assume (without loss of generality) that elements are indexed such that $x_i^j <_j x_k^j$ iff $i < k$. We also assume that the output space $\mathscr{Y}$ is ordered and we note $\leq_{\mathscr{Y}}$ the order on $\mathscr{Y}$, assuming an indexing such that $y_i <_{\mathscr{Y}} y_k$ iff $i < k$. Given two elements $x, y \in \mathscr{X}^{1:D}$, we simply write $x \geq y$ if $x^j \geq_j y^j$ for $j = 1, \ldots, n$, and $x < y$ if moreover $x \neq y$ (i.e., $x^j <_j y^j$ for at least one $j$).

We assume that the function is non-decreasing in each of its arguments $X^j$, that is

$$\phi(x_{i_1}^1, \ldots, x_{i_\ell}^\ell, \ldots, x_{i_D}^D) \leq_{\mathscr{Y}} \phi(x_{i_1}^1, \ldots, x_{i_\ell'}^\ell, \ldots, x_{i_D}^D) \quad (10)$$

iff $i_\ell \leq i_\ell'$. Note that a function monotone in each variable $X^j$ can always be transformed into a non-decreasing one, simply by reversing $\leq_j$ for those variables $X^j$ in which $\phi$ is non-increasing.

We now consider the problem of estimating the uncertainty of some event $\{\phi(\cdot) \geq d\}$ (or $\{\phi(\cdot) < d\}$, obtained by duality). Evaluating the uncertainty over such events is instrumental in a number of applications, such as risk analysis [2]. Given a value $d \in \mathscr{Y}$, let us define the concept of minimal path and minimal cut vectors.

**Definition 1.** A minimal path (MP) vector $x$ of function $\phi$ for value $d$ is an element $x \in \mathscr{X}^{1:D}$ such that $\phi(x) \geq d$ and $\phi(y) < d$ for any $x > y$ ($x$ is a minimal element in $\{x : \phi(x) \geq d\}$).

**Definition 2.** A minimal cut (MC) vector $x$ of function $\phi$ for value $d$ is an element $x \in \mathscr{X}^{1:D}$ such that $\phi(x) < d$ and $\phi(y) \geq d$ for any $x < y$ ($x$ is a maximal element in $\{x : \phi(x) < d\}$).

Let $p_1, \ldots, p_P$ be the set of all minimal path vectors of some function for a given performance level $d$ (means to obtain minimal paths are provided by Xue [18]). We note $A_{p_i} = \{x \in \mathscr{X}^{1:D} | x \geq p_i\}$ the set of configurations dominating the minimal path vector $p_i$ and $\mathscr{A}_{\mathscr{P}} = \{A_{p_1}, \ldots, A_{p_P}\}$ the set of events induced by minimal path vectors. Note that

$$A_{p_i} = \times_{j=1}^D \{x^j | x^j \geq_j p_i^j\} \quad (11)$$

is rectangular, hence we can use results from Section 3.

**Lemma 4.** *The rectangular sets $\mathscr{A}_{\mathscr{P}}$ induced by minimal path vectors satisfy Theorem 5*

*Proof.* Consider two minimal path vectors $A_{p_i}, A_{p_j}$ and a dimension $\ell$, then either $\{x^\ell \geq_\ell p_i^\ell\} \subseteq \{x^\ell \geq_\ell p_j^\ell\}$ or $\{x^\ell \geq_\ell p_i^\ell\} \supseteq \{x^\ell \geq_\ell p_j^\ell\}$. $\qquad\square$

It can be checked that $\{x \in \mathscr{X}^{1:D} | \phi(x) \geq d\} = \cup_{i=1}^P A_{p_i}$. We can therefore write the inclusion/exclusion formula for belief functions:

$$
\begin{aligned}
Bel(\phi(x) \geq d) &= Bel(A_{p_1} \cup \ldots \cup A_{p_P}) \\
&= \sum_{\mathscr{I} \subseteq \mathscr{A}_{\mathscr{P}}} (-1)^{|\mathscr{I}|+1} Bel(\cap_{A \in \mathscr{I}} A), \\
&= 1 - Pl(\phi(x) < d) \quad (12)
\end{aligned}
$$

Under the hypothesis of random set independence, computing each term simplifies into

$$Bel(A_{p_j} \cap \ldots \cap A_{p_k}) = \prod_{i=1}^D Bel(\{x^i \geq \max\{p_j^i, \ldots, p_k^i\})$$

The computation of $Bel(\phi(x) < d)$ can be done similarly by using minimal cut vectors. Let $\mathscr{C}_1, \ldots, \mathscr{C}_C$ be the set of all minimal cut vectors of $\phi$. Then $A_{\mathscr{C}_i} = \{x \in \mathscr{X}^{1:D} | x \leq \mathscr{C}_i\} = \times_{j=1}^D \{x^j | x^j \leq_j \mathscr{C}_i^j\}$ is rectangular and we have the following result, whose proof is similar to the one of Lemma 4.

**Lemma 5.** *The rectangular sets $\mathscr{A}_{\mathscr{C}}$ induced by minimal cut vectors satisfy Theorem 5*

Denoting by $\mathscr{A}_{\mathscr{C}} = \{A_{\mathscr{C}_1}, \ldots, A_{\mathscr{C}_C}\}$ the set of events induced by minimal cut vectors, we have that $\{x \in \mathscr{X}^{1:D} | \phi(x) < d\} = \cup_{i=1}^C A_{\mathscr{C}_i}$, hence applying the inclusion/exclusion formula for belief functions gives

$$
\begin{aligned}
Bel(\phi(x) < d) &= Bel(A_{\mathscr{C}_1} \cup \ldots \cup A_{\mathscr{C}_C}) \\
&= \sum_{\mathscr{I} \subseteq \mathscr{A}_{\mathscr{C}}} (-1)^{|\mathscr{I}|+1} Bel(\cap_{A \in \mathscr{I}} A), \\
&= 1 - Pl(\phi(x) \geq d). \quad (13)
\end{aligned}
$$

Let us now illustrate how this result can be applied to reliability problems.

## 4.2 Application to Multi-State Systems (MSS) reliability

Using the inclusion/excusion formula is a classical way of estimating system reliability. In this section we show that, thanks to our results, we can extend it to the case where system components can be in multiple states and where the uncertainty about these states is given by belief functions. We refer to Lisnianski and Levitin [15] for a detailed review of the problem.

MSS analysed in this section are such that

- their components are s-independent, meaning that the state of one component has no influence over the state of other components;

- the states of each component are mutually exclusive;

- the MSS is coherent (if one state component efficiency increases, the overall efficiency increases).

Let us now show that for such systems, we can define minimal path sets and minimal cut sets that satisfy the exclusion/inclusion principle.

In reliability analysis, variables $X^j$, $j = 1, \ldots, D$ correspond to the $D$ components of the system and the value $x_i^j$ is the $i$th state of component $j$. Usually, states are ordered according to their performance rates, hence we can assume the spaces $\mathscr{X}^j$ to be ordered. $\mathscr{X}^{1:D}$ corresponds to the system states and $\mathscr{Y} = \{y_1, \ldots, y_Y\}$ is the ordered set of global performance rates of the system.

The structure function $\phi : \mathscr{X}^{1:D} \to \mathscr{Y}$ links the system states to its global performance. As the system is coherent, function $\phi$ is non-decreasing, in the sense of Eq. (10).

As a typical task in multi-state reliability analysis is to estimate with which certainty a system will guarantee a level $d$ of performance, results from Section 4.1 directly apply.

*Example* 1. Let us now illustrate our approach on a complete example, inspired from Ding and Lisnianski [10].

In this example, we aim to evaluate the availability of a flow transmission system design presented in Fig. 3 and made of three pipes. The flow is transmitted from left to right and the performance levels of the pipes are measured by their transmission capacity (tons of per minute). It is supposed that elements 1 and 2 have three states: a state of total failure corresponding to a capacity of 0, a state of full capacity and a state of partial failure. Element 3 only has two states: a state of total failure and a state of full capacity. All performance levels are precise.

The state performance levels and the state probabilities of the flow transmitter system are given in Table 2. These probabilities could have been obtained the imprecise Dirichlet model [4] considered in Li *et al.* [14]. We
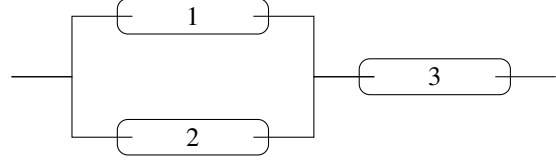


Figure 3: Flow transmission system

aim to estimate the availability of the system when $d = 1.5$. The minimal paths are

$$p_1 = (x_1^1, x_2^2, x_3^3) = (0, 1.5, 4), \ p_2 = (x_3^1, x_1^2, x_3^3) = (1.5, 0, 4).$$

The set $A_{p_1}$ and $A_{p_2}$ of vectors $a$ such that $a \geq p_1$, $b \geq p_2$ are

$$A_{p_1} = \{0, 1, 1.5\} \times \{1.5, 2\} \times \{4\} \text{ and}$$
$$A_{p_2} = \{1.5\} \times \{0, 1.5, 2\} \times \{4\},$$

and their intersection $A_{p_1} \cap A_{p_2}$ consists of vectors $c$ such that $c \geq p_1 \vee p_2$ (with $\vee = \max$), that is:

$$A_{p_1} \cap A_{p_2} = \{1.5\} \times \{1.5, 2\} \times \{4\}.$$

Applying the inclusion/exclusion formula for a requested level $d = 1.5$, we obtain

$$Bel(\phi \geq 1.5) \quad = Bel(A_{p_1}) + Bel(A_{p_2}) - Bel(A_{p_1} \cap A_{p_2})$$

For example, we have

$$
\begin{aligned}
Bel(A_{p_1}) &= Bel(\{0, 1, 1.5\} \times \{1.5, 2\} \times \{4\}) \\
&= Bel(\{0, 1, 1.5\}).Bel(\{1.5, 2\}).Bel(\{4\}) \\
&= 1 * 0.895 * 0.958 \\
&= 0.8574
\end{aligned}
$$

and $Bel(A_{p_2})$, $Bel(A_{p_1} \cap A_{p_2})$ can be computed similarly. Finally we get

$$Bel(\phi \geq 1.5) \quad = \quad 0.8574 + 0.7654 - 0.6851 = 0.9377$$

and by duality with $Bel(\phi < 1.5)$, we get

$$Pl(\phi \geq 1.5) \quad = \quad 1 - Bel(\phi < 1.5) = 0.9523.$$

The availability $A_s$ of the flow transmission system for a requested performance level $d = 1.5$ is given by $[Bel(A), Pl(A)] = [0.9377, 0.9523]$.

## 5 The case of Boolean formulas

In this section, we consider binary spaces $\mathscr{X}^i$, and lay bare conditions for applying the inclusion/exclusion property to Boolean formulas expressed in Disjunctive Normal Form (DNF).

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x^1$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 |
| $x^2$ | 0 | 0 | 1.5 | 1.5 | 2 | 2 | 0 | 0 | 1.5 | 1.5 | 2 | 2 | 0 | 0 | 1.5 | 1.5 | 2 | 2 |
| $x^3$ | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 |
| $y = \Phi(x^1,x^2,x^3)$ | 0 | 0 | 0 | 1.5 | 0 | 2 | 0 | 1 | 0 | 2.5 | 0 | 3 | 0 | 1.5 | 0 | 3 | 0 | 3.5 |

Table 1: Performance rates of the oil transmission system

| $X^j$ | 1 | 2 | 3 |
|---|---|---|---|
| $p_1^j$ | [0.096,0.106] | [0.095,0.105] | - |
| $p_2^j$ | [0.095,0.105] | [0.195,0.205] | [0.032,0.042] |
| $p_3^j$ | [0.799,0.809] | [0.7,0.71] | [0.958,0.968] |
| $g_1^j$ | 0 | 0 | - |
| $g_2^j$ | 1 | 1.5 | 0 |
| $g_3^j$ | 1.5 | 2 | 4 |

Table 2: Parameters of the flow transmission system

In propositional logic, each $\mathcal{X}^i = \{x^i, \bar{x}^i\}$ can be associated to a variable also denoted by $x_i$, and $\mathcal{X}^{1:D}$ is the set of interpretations of the propositional language generated by the set $\mathcal{V}$ of variables $x_i$. In this case, $x^i$ is understood as an atomic proposition, while $\bar{x}^i$ denotes its negation. Any rectangular set $A \subseteq \mathcal{X}^{1:D}$ can then be interpreted as a conjunction of literals (often called a partial model), and given a collection of $n$ such partial models $\mathcal{A}_n = \{A_1, \ldots, A_n\}$, the event $A_1 \cup \ldots \cup A_n$ is a Boolean formula expressed in Disjunctive Normal Form (DNF - a disjunction of conjunctions). All Boolean formulas can be written in such a form.

A convenient representation of a partial model $A$ is in the form of an orthopair [6] $(P,N)$ of disjoint subsets of indices of variables $P, N \subset [1,D]$ such that $A_{(P,N)} = \bigwedge_{k \in P} x^k \wedge \bigwedge_{k \in N} \bar{x}^k$. Then a singleton in $\mathcal{X}^{1:D}$ is of the form $\bigwedge_{k \in P} x^k \wedge \bigwedge_{k \in \bar{P}} \bar{x}^k$, i.e. corresponds to an orthopair $(P, \bar{P})$.

We consider that the uncertainty over each Boolean variable $x^i$ is described by a belief function $Bel^i$. For simplicity, we shall use $x^i$ as short for $\{x^i\}$ in the argument of set-functions. As $\mathcal{X}^i$ is binary, its mass function $m^i$ only needs two numbers to be defined, e.g., $l^i = Bel^i(x^i)$ and $u^i = Pl^i(x^i)$. Indeed, we have $Bel^i(x^i) = l^i = m^i(x^i)$, $Pl^i(x^i) = 1 - Bel^i(\bar{x}^i) = 1 - m^i(\bar{x}^i)$ and $m^i(\mathcal{X}^i) = u^i - l^i$. For $D$ marginal masses $m^i$ on $\mathcal{X}^i$, $i = 1, \ldots, D$, the joint mass $m$ on $\mathcal{X}^{1:D}$ can be computed as follows for any partial model $A_{(P,N)}$, applying Equation(8):

$$m(A_{(P,N)}) = \prod_{i \in P} l^i \prod_{i \in N}(1 - u^i) \prod_{i \notin P \cup N}(u^i - l^i) \qquad (14)$$

We can particularize Theorem 5 to the case of Boolean formulas, and identify conditions under which the belief or the plausibility of a DNF can be easily estimated using Equality (1), changing probability into belief. Let us see how the conditions exhibited in this theorem can be expressed in the Boolean case.

Consider the first condition of Theorem 5

$$\exists p \neq q \in \{1, \ldots, D\} \text{ such that } A_i^p \cap A_j^p = A_i^q \cap A_j^q = \emptyset.$$

Note that when spaces are binary, $A_i^p = x^p$ (if $p \in P_i$), or $A_i^p = \bar{x}^p$ (if $p \in N_i$), or yet $A_i^p = \mathcal{X}^i$ (if $p \notin P_i \cup N_i$). $A_i \cap A_j = \emptyset$ therefore means that for some index $p$, $p \in (P_i \cap N_j) \cup (P_j \cap N_i)$ (there are two opposite literals in the conjunction).

The condition can thus be rewritten as follows, using orthopairs $(P_i, N_i)$ and $(P_j, N_j)$:

$$\exists p \neq q \in \{1, \ldots, D\} \text{ such that } p, q \in (P_i \cap N_j) \cup (P_j \cap N_i).$$

For instance, consider the equivalence connective $x^1 \iff x^2 = (x^1 \wedge x^2) \vee (\bar{x}^1 \wedge \bar{x}^2)$ so that $A_1 = x^1 \wedge x^2$ and $A_2 = \bar{x}^1 \wedge \bar{x}^2$. We have $p = 1 \in P_1 \cap N_2, q = 2 \in P_1 \cap N_2$, hence the condition is satisfied and $Bel(x^1 \iff x^2) = Bel(x^1 \wedge x^2) + Bel(\bar{x}^1 \wedge \bar{x}^2)$ (the remaining term is $Bel(\emptyset)$).

The second condition of Theorem 5 reads

$$\forall \ell \in \{1, \ldots, D\} \text{ either } A_i^\ell \subseteq A_j^\ell \text{ or } A_j^\ell \subseteq A_i^\ell$$

and the condition $A_i^\ell \subseteq A_j^\ell$ can be expressed in the Boolean case as:

$$\ell \in (P_i \cap \bar{N}_j) \cup (N_i \cap \bar{P}_j) \cup (\bar{P}_i \cap \bar{N}_i \cap \bar{P}_j \cap \bar{N}_j).$$

The condition can thus be rewritten as follows, using orthopairs $(P_i, N_i)$ and $(P_j, N_j)$:

$$P_i \cap N_j = \emptyset \text{ and } P_j \cap N_i = \emptyset$$

For instance consider the disjunction $x^1 \vee x^2$, where $A_1 = x^1$ and $A_2 = x^2$, so that $P_1 = \{1\}, P_2 = \{2\}, N_1 = N_2 = \emptyset$. So $Bel(x^1 \vee x^2) = Bel(x^1) + Bel(x^2) - Bel(x^1 \wedge x^2)$.

We can summarize the above results as

**Proposition 6.** *The set of partial models $\mathcal{A}_n = \{A_1, \ldots, A_n\}$ satisfies the inclusion/exclusion principle if and only if, for any pair $A_i, A_j$ one of the two following conditions is satisfied:*

- $\exists p \neq q \in \{1, \ldots, D\}$ *s.t.* $p, q \in (P_i \cap N_j) \cup (P_j \cap N_i)$.

- $P_i \cap N_j = \emptyset$ *and* $P_j \cap N_i = \emptyset$

This condition tells us that for any pair of partial models, :

- either conjunctions $A_i, A_j$ contain at least two opposite literals,

- or events $A_i, A_j$ have a non-empty intersection and have a common model.

These conditions allow us to check, once a formula has been put in DNF, whether or not the inclusion/exclusion principle applies. Important particular cases where it applies are disjunctions of partial models having only positive (negative) literals, of the form $A_1 \cup \ldots \cup A_n$, where $N_1 = \ldots = N_n = \emptyset$ ($P_1 = \ldots = P_n = \emptyset$). This is the typical Boolean formula one obtains in fault tree analysis, where the system failure is due to the failures of some subsets of components, the latter failures being modelled by positive literals. More generally, the inclusion/exclusion principle applies to disjunctions of partial models which can, via a renaming, be rewritten as a disjunction of conjunctions of positive literals: namely, whenever a single variable never appears in a positive and negative form in two of the conjunctions.

As an example where the inclusion/exclusion principle cannot be applied, consider the formula $x^1 \vee (\bar{x}^1 \wedge x^2)$ (which is just the disjunction $x^1 \vee x^2$ we already considered above). It does not hold that $Bel(x^1 \vee (\bar{x}^1 \wedge x^2) = Bel(x^1) + Bel(\bar{x}^1 \wedge x^2)$, since the latter sum neglects $m(x^2)$, where $x^2$ is a focal set that implies neither $x^1$ nor $\bar{x}^1 \wedge x^2$. Note that this remark suggests that normal forms that are very useful to compute the probability of a Boolean formula efficiently may not be useful to speed up computations of belief and plausibility degrees. For instance, $x^1 \vee (\bar{x}^1 \wedge x^2)$ is a binary decision diagram (BDD) [5] for the disjunction, and this form prevents $Bel(x^1 \vee x^2)$ from being computed using the inclusion/exclusion principle.

We can give explicit expressions for the belief and plausibility of conjunctions or disjunctions of literals in terms of marginal mass functions:

**Proposition 7.** *The belief of a conjunction $C_{(P,N)} = \bigwedge_{k \in P} x^k \wedge \bigwedge_{k \in N} \bar{x}^k$, and that of a disjunction $D_{(P,N)} = \bigvee_{k \in P} x^k \vee \bigvee_{k \in N} \bar{x}^k$ of literals forming an orthopair $(P,N)$ are respectively given by:*

$$Bel(C_{(P,N)}) = \prod_{i \in P} l^i \prod_{i \in N}(1 - u^i), \qquad (15)$$

$$Bel(D_{(P,N)}) = 1 - \prod_{i \in P}(1 - l^i) \prod_{i \in N} u^i. \qquad (16)$$

*Proof.* $Bel(C_{(P,N)})$ can be obtained by applying Equation (9) to $C_{(P,N)}$.

For $Bel(D_{(P,N)})$, we have

$$
\begin{aligned}
Pl(C_{(N,P)}) &= Pl(\wedge_{i \in N} x^i \wedge \wedge_{i \in P} \bar{x}^i) \\
&= \prod_{i \in N}(1 - l^i) \prod_{i \in P} u^i \\
&= 1 - (1 - \prod_{i \in N}(1 - l_i) \prod_{i \in P} u_i) \\
&= 1 - Bel(\vee_{i \in N} \bar{x}^i \vee \vee_{i \in P} x^i) \\
&= 1 - Bel(D_{(P,N)})
\end{aligned}
$$

where the second equality following from Equation (9). □

Using the fact that $Bel(C_{(N,P)}) = 1 - Pl(D_{(P,N)})$, we can deduce

$$Pl(D_{(P,N)}) = 1 - \prod_{i \in P} l^i \prod_{i \in N}(1 - u^i). \qquad (17)$$

$$Pl(C_{(P,N)}) = \prod_{i \in P} u^i \prod_{i \in N}(1 - l^i). \qquad (18)$$

To compute the plausibility of a formula $\phi$, we can put it in conjunctive normal form, that is as a conjunction of clauses $\wedge_{i=1}^k \kappa_i$ where the $\kappa_i$'s are disjunctions of literals. Then we can write:

$$Pl(\phi) = 1 - Bel(\neg(\wedge_{i=1}^k \kappa_i)) = 1 - Bel(\vee_{i=1}^k \neg \kappa_i) \quad (19)$$

Noticing that the terms $\neg \kappa_i$ are rectangular (partial models), we can apply Proposition 6 again (this trick can be viewed as an application of results of Subsection 4.1 to ordered scale $\mathscr{X} = \{0 < 1\}$). As a consequence we can compute the belief and the plausibility of any logical formula that obeys the conditions of Proposition 6 in terms of the belief and plausibilities of atoms $x^i$.

*Example* 2. For instance consider the formula $\phi = (x^1 \wedge \bar{x}^2) \vee (\bar{x}^1 \wedge x^2) \vee x^3$, with $A_1 = x^1 \wedge \bar{x}^2, A_2 = \bar{x}^1 \wedge x^2, A_3 = x^3$. It satisfies Proposition 6, and

$Bel(\phi) = Bel(x^1 \wedge \bar{x}^2) + Bel(\bar{x}^1 \wedge x^2)$
$+ Bel(x^3) - Bel(x^1 \wedge \bar{x}^2 \wedge x^3) - Bel(\bar{x}^1 \wedge x^2 \wedge x^3)$
$= l_1(1 - u_2) + (1 - u_1)l_2 + l_3(1 - l_1(1 - u_2) - (1 - u_1)l_2)$

In CNF, this formula reads : $(x^1 \vee x^2) \wedge (\bar{x}^1 \vee \bar{x}^2) \wedge x^3$. Then:

$Pl(\phi) = 1 - Bel((x^1 \wedge x^2) \vee (\bar{x}^1 \wedge \bar{x}^2) \vee \bar{x}^3);$
$= 1 - Bel(x^1 \wedge x^2) - Bel(\bar{x}^1 \wedge \bar{x}^2) - Bel(\bar{x}^3)$
$+ Bel(x^1 \wedge x^2 \wedge \bar{x}^3) + Bel(\bar{x}^1 \wedge \bar{x}^2 \wedge \bar{x}^3)$
$= 1 - l^1 l^2 - (1 - u^1)(1 - u^2) - 1 + u^3 + l_1 l_2(1 - u^3)$
$+ (1 - u^1)(1 - u^2)(1 - u^3)$

## 6 Conclusion

We provided necessary and sufficient conditions for the inclusion/exclusion principle to hold with belief functions.

To demonstrate the usefulness of those results, we discussed their application to system reliability and to uncertainty evaluation over DNF and CNF Boolean formulas.

We can mention several lines of research that would complement the present results: (1) find necessary and sufficient conditions for the inclusion/exclusion principle to hold for plausibilities in the general case (a counterpart to Proposition 5); (2) investigate the relation between the assumption of random set independence (made in this paper) and other types of independence [12]; (3) investigate how to decompose an event / a formula into a set of event satisfying the inclusion/exclusion principle (e.g., classical BDDs do not always provide adequate solutions).

## Acknowledgements

## References

[1] C. Baudrit and D. Dubois. Comparing methods for joint objective and subjective uncertainty propagation with an example in a risk assessment. In *Proc. Fourth International Symposium on Imprecise Probabilities and Their Application (ISIPTA'05)*, pages 31–40, Pittsburg (USA, Pennsylvanie), 2005.

[2] C. Baudrit, D. Guyonnet, and D. Dubois. Joint propagation and exploitation of probabilistic and possibilistic information in risk assessment. *IEEE Trans. Fuzzy Systems*, 14:593–608, 2006.

[3] Y. Ben-Haim. *Info-gap decision theory: decisions under severe uncertainty*. Academic Press, 2006.

[4] J.M. Bernard. An introduction to the imprecise dirichlet model for multinomial data. *International Journal of Approximate Reasoning*, 39(2):123–150, 2005.

[5] R.E. Bryant. Symbolic boolean manipulation with ordered binary-decision diagrams. *ACM Computing Surveys (CSUR)*, 24(3):293–318, 1992.

[6] D. Ciucci. Orthopairs: A simple and widely used-way to model uncertainty. *Fundam. Inform.*, 108(3-4):287–304, 2011.

[7] I. Couso, S. Moral, and P. Walley. A survey of concepts of independence for imprecise probabilities. *Risk Decision and Policy*, 5:165–181, 2000.

[8] A.P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, 38:325–339, 1967.

[9] S. Destercke. Independence and 2-monotonicity: Nice to have, hard to keep. *International Journal of Approximate Reasoning (In press)*, 2012.

[10] Y. Ding, M. J. Zuo, A. Lisnianski, and Z. G. Tian. Fuzzy multi-state system: General definition and performance assessment. *IEEE Transactions on Reliability*, 57:589 – 594, 2008.

[11] D. Dubois and H. Prade. *Possibility Theory: An Approach to Computerized Processing of Uncertainty*. Plenum Press, New York, 1988.

[12] C. Jacob, D. Dubois, and J. Cardoso. Evaluating the uncertainty of a boolean formula with belief functions. *Advances in Computational Intelligence*, pages 521–531, 2012.

[13] E.T. Jaynes. *Probability Theory: The Logic of Science*. Cambridge University Press, 2003.

[14] C-Y. Li, X. Chen, X-S. Yi, and J y. Tao. Interval-valued reliability analysis of multi-state systems. *IEEE Transactions on Reliability*, 60:323 – 330, 2011.

[15] A. Lisnianski and G. Levitin. M*ulti-*St*ate* S*ystem Reliability: Assessment, Optimization and Applications*. World Scientific Publishing Co Pte Ltd, 2003.

[16] G. Shafer. *A mathematical Theory of Evidence*. Princeton University Press, New Jersey, 1976.

[17] P. Walley. *Statistical reasoning with imprecise Probabilities*. Chapman and Hall, New York, 1991.

[18] J. Xue. On multistate system analysis. *IEEE Transactions on Reliability*, pages 329–337, 1985.